Some Applications of Fractional Calculus in Elementary Physics

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*The article is dedicated with immense love and affectionate regards to very dear Shruti, Shreya, Vibhuti, Sujata Ahuja, and Rajiv Ahuja.

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Abstract

A brief introduction to development of fractional calculus, particularly with time as independent variable, and its importance in taking care of memory effects has been given. As expository applications of this, topics of Newton's second law of motion and equations of motion, a body falling under gravity in a viscous medium, projectile motion in a resistive medium, simple oscillator, Fourier law of heat conduction, Newton's law of cooling, Ohm's law and RC, RL circuits, and physics of decay / growth / relaxation processes, have been dealt with. The objective of this article is not only to put forth the well-established things from introductory physics in different perspective but also to bring out the need to include the old (almost as much as the classical calculus) and yet offbeat topic of fractional calculus as a part of our curriculum at some stage, and, thus, prepare the next generation to capture the richness underlying the complex systems in nature.

1 Introduction

In day-to-day life, we often come across situations where a change or variation in one quantity (called independent variable in mathematics) brings about a change in another related quantity (dependent variable) and we are interested in finding the relevant rate of change for a specific value of the former. Some typical examples where change (increase) in time produces a change in an observed quantity are: position of a moving person or a vehicle, leakage of water from a tank through a hole, melting of ice cubes, glucose infused to a patient, growth or decay of some species, and so on. It is not necessary that the independent variable be time. It can be any other quantity as well. For example, the description of trajectory of projectiles like a cricket ball hit for a six, a long jumper during flight, a missile propelled at a target, etc. involves height as dependent variable and its projection on the ground, with respect to the starting point, as independent variable. Similarly, the effect of a change in the price of a product on its demand has price as an independent variable. From mathematics point of view, study of the rates of change falls under the purview of differential calculus.

To be more explicit, differential calculus is branch of mathematics that deals with precise description of rates and their wide-range applications in different disciplines of science and engineering. This rate of variation, in the limit of infinitesimally small change in the independent variable at the chosen input value, is called derivative of the dependent variable (function) for that value. Geometrically speaking, the derivative for a specific value of independent variable is the slope of the tangent line to the graph of the function at that point, provided that the derivative exists and is defined there. Furthermore, treating the derivative so obtained as next function, we get second-order derivative, and a repetition of this process leads to higher-order derivatives. For x as independent variable and $y(x)$ as dependent variable, the nth-order derivative is denoted by $y^{(n)}(x)$ or $\frac{d^n y(x)}{dx^n}$, which for time-dependent function $f(t)$ reads $f^{(n)}(t)$ or $\frac{d^n f(t)}{dt^n}$. In this notation, $y^{(0)}(x)$ and $f^{(0)}(t)$ denote the original function itself. Sometimes, symbol D^n, where D denotes operator $\frac{d}{dx}$ or $\frac{d}{dt}$ which when applied to a function yields its derivative with respect to the independent variable, is also used to represent nth-order differentiation. Obviously, $D^0 f(t) = f(t)$. As typical examples, we may mention that the first-order time-derivatives $Df(t)$ for $f(t) = constant(C)$, t^j, $\sin(\omega t)$, e^{at} are zero, jt^{j-1}, $\omega\cos(\omega t)$, ae^{at}, respectively. Similarly, the second-order derivatives $D^2 f(t)$ of the last three functions are $j(j-1)t^{j-2}$, $-\omega^2\sin(\omega t)$, $a^2 e^{at}$. Since the functions so obtained are evaluated for a specific value of t, these derivatives give instantaneous rates of change in $f(t)$, and are localized in nature getting no contribution from the earlier values.

It is interesting to note that information about the derivative (say, $\frac{dF(t)}{dt} \equiv f(t)$), can be used to obtain the relevant function $F(t)$ uncertain to a constant (because derivative of a constant is zero). This process of determining the antiderivative $D^{-1}f(t)$, written as $F(t) = \int f(t)dt$, is known as integration, and $F(t)$ is an indefinite integral of $f(t) = F^{(1)}(t)$ called the integrand. This process too can be carried out repeatedly to get $F(t)$ from $F^{(n)}(t)$. The indefinite integrals of $f(t) = C, t^j, \sin(\omega t), e^{at}$ are $Ct + C_1$, $\frac{t^{j+1}}{j+1} + C_2$, $-\frac{1}{\omega}\cos(\omega t) + C_3$, $\frac{e^{at}}{a} + C_4$, respectively. Note that if the integral is found over a specific interval of values of the independent variable t from $t = t_l$ to $t = t_u$, then we get the definite integral $I = \int_{t_l}^{t_u} f(t)dt$, which depends not only only on the nature of $f(t)$ but also on the values of upper and lower limits t_u and t_l. The value of I is determined by evaluating the indefinite integral $\int f(t)dt$, and then finding the difference of its values for t_u and t_l; i.e., $I = F(t_u) - F(t_l)$. In geometrical sense, I represents sum of areas (in the general sense) of a large number N of adjoining small strips of length $f(t)$ and infinitesimal width $dt = \frac{t_u - t_l}{N}$, formed by plotting $f(t)$ as a function of t over the chosen interval t_l to t_u. Obviously, the definite integral I incorporates complete information about the behaviour of $f(t)$ from t_l to t_u. The study of various aspects of integrals and their evaluation forms the content of integral calculus.

The subjects of differential calculus and integral calculus together constitute calculus. The credit for invention and initial development of this branch of mathematics goes to Leibniz and Newton, (independently of each other, first publishing their works around the same time in the late 17th century). Leibniz started with integration and directed his efforts on developing the formalism in proper perspective and defining appropriate symbols for different concepts. On the other hand, Newton started with differentiation and put emphasis on the use of the rules framed for applications in physics. However, it may be mentioned that prior to its formal development in the seventeenth century and after that, some ideas of calculus were used by scientists in different countries including India for specific purposes. Today, calculus finds widespread applications in almost all branches of science, engineering, and economics.

It is worth pointing out that n, defining the order of derivative, has integer values. It is natural to ask: What will be outcome if n has a non-integer value? In fact, such a question was posed by L'Hospital, one of the prominent developers of calculus, to Leibniz, nearly 327 years back in 1695, when he sought the latter's opinion about the meaning of $\frac{d^n}{dx^n}$ if n were $\frac{1}{2}$. It was on Sept 30, 1695, that Leibniz replied that 'this was an apparent paradox', and that one day, useful consequences would be drawn from this'. It was followed by occasional reference to fractional derivatives (FDs) by some mathematicians till 1819, when Lacroix showed in his book that $\frac{d^{1/2}x}{dx^{1/2}} = 2\sqrt{\frac{x}{\pi}}$. In 1823, 21 years old Abel, while solving the tautochrone problem (viz., determination of a curve in the vertical plane such that the time taken by a particle to slide down without friction under the influence of uniform gravity to its lowest point is independent of its initial position on the curve), put forward the essential structure of fractional calculus (FC), and introduced the FD in the form now named after Caputo. This work drew attention of Liouville, who made serious effort to develop the subject in a logical manner from 1832 to 1837 and applied his formulae to solve problems in electromagnetism. Besides, these creative minds, prominent contributors to the development of this subject until the middle of 20th century included brilliant mathematicians Riemann, Grunwald, Letnikov, Sonin, Weyl, Erdelyi, and numerous others. Interestingly, many of them introduced their own formulae for fractional (and even generalized) derivative and integral, with strange properties, following different approaches. However, some of the early definitions of FDs had a limited scope and could not be considered as general definitions, and that trend of suggesting new definitions continues to be there even now.

A significant development from the point-of-view of future applications of the FC occurred in 1967 when Caputo put forward his definition of FD by reformulating the Riemann-Liouville formula to accommodate the initial conditions (for fractional differential equations) as are done in the corresponding problems based on integer-order derivatives. However, a systematic interest in FC and its applications, started only in 1974 when first exclusively dedicated international conference was organized by Ross at University of New Haven (USA) and a well-written book, detailing concepts and techniques of the subject and their application to problems in physics, chemistry, and engineering, was brought out by Oldham (a chemist) and Spanier (a mathematician). Since then, holding conferences, publishing books as well as international research journals devoted to this subject and its various aspects, has become a regular activity for the scientists and engineers working in this field. The main reason for a gap of nearly 280 years in the birth of the subject and its applications is that unlike integer-order derivative there are numerous definitions of FD and that a simple geometrical interpretation was missing. Interestingly, various definitions of FD reduce to the standard derivative for integer value of the order but might not be equivalent for the non-integer order.

As it stands now, FC has emerged as a branch of mathematics that deals with integrals and derivatives of any arbitrary real non-integer or complex order α; α may even be a complex function of time and space coordinates. Despite this generalization of the definition of order, the name still contains word fractional for historical reasons. As will be seen in section 3, the main feature of FD of a function, which makes it different from the integer-order derivative, is that it incorporates the contribution of nonlocal effects to the dependent variable with respect to changes in space and time. In the context of time, this means that the FD at any instant of time also includes the influence of the past behaviour of the function as if it had some memory to remember the states through which it had passed. Of course, it must be emphasized that the system described by such a function does not have any hidden intelligence but only delayed effects of collisions, interactions, causative forces, etc. In addition to this, fractional vector calculus involving FDs with respect to space coordinates has also been developed.

Besides their spectacular use in the development of classical mechanics (Newtonian, and the latter sophisticated Lagrangian and Hamiltonian formulations) and electromagnetics, the concepts and methods of FC provide a powerful mathematical tool for an elegant description of problems involving power-law non-locality and memory effects leading to better insight into the nature of the systems. These include anomalous diffusion in porous media, dense polymer solutions, composite heterogeneous films, etc.; the behaviour of viscoelastic materials, colloidal systems, magneto-rheological fluids, and amorphous semiconductors; wave propagation in media with long-range interaction; statistical mechanics of systems with long-range power-law interactions; electrodynamics of systems characterized by nonlocal dielectric properties of the media; frequency dependent acoustic wave propagation in porous media; signal and image processing; analysis and synthesis of different types of control systems; models for description of environment; and mathematical modelling of economic processes and population growth with memory effects.

It is important to note that Heisenberg's uncertainty principle and tunneling effect in quantum mechanics are essentially manifestation of nonlocality implying that this aspect is inherent in quantum mechanics (QM). Accordingly, a lot of effort has been directed at developing nonrelativistic as well as relativistic QM, quantum field theory, and statistical mechanics in the framework of FC and wide-ranging applications of these in atomic, molecular, nuclear, particle, and metal cluster physics.

FC is also being used as an invaluable technique in the description of physics of biological structures and living organisms. Some topics in this category are DNA dynamics; protein folding; and modeling for neuron activity, for cancer growth, for HIV dynamics, for human autoimmune diseases, and for transport of drugs. Very recently, some work has been reported on developing FC based models for understanding dynamics of covid-19.

Generally, the reformulation of conventional physics problems in the framework of FC is carried out either by replacing the time-derivatives of integer order in the mathematical description by the relevant fractional-order derivative or by introducing FDs in the expression for the force field. The first approach is followed in the case of classical mechanics, the systems described by classical wave equation or by Schrodinger equation, etc. On the other hand, the second option is used in handling problems in field theory, nuclear physics, etc.

The purpose of the present article is to delineate upon the basics of FC in terms of fractional-order temporal derivatives and related mathematical tools needed for handling simple problems. This aspect is then illustrated by derivation and discussion of formulae for different topics of pedagogical importance from elementary physics. These are: Newton's second law of motion and equations of motion, a body falling under gravity in a viscous medium, projectile motion, simple oscillator, Fourier law of thermal conduction, Newton's law of cooling, Ohm's law and RC, RL circuits, and physics of decay / growth / relaxation processes [1-13].

However, before embarking upon this aspect, we briefly dwell upon some special functions of mathematics that play an important role in the formulation and applications of FC.

2 Some Relevant Special Functions

2.1 Factorial and Gamma Functions

The factorial of a positive integer n, denoted by $n!$, is the product of all the integers from 1 to that number, i.e.,

$$n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n = \prod_{j=1}^{n} j. \tag{1}$$

By convention, $0! = 1$. Obviously, $n! = (n-1)! \times n$. An extension of the factorial, which allows n to have non-integer or even complex values, is provided by Euler's gamma function. For a complex number z, such that its real part is positive, this function is defined, in integral form, as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{dt}. \tag{2}$$

Integrating by parts, we get

$$\Gamma(z) = (z-1)\Gamma(z-1), \tag{3}$$

implying that $\Gamma(z+1) = z\Gamma(z)$. Also, from Eq. (2), we have $\Gamma(1) = \int_0^\infty e^{-t}\mathrm{dt} = 1$. This together with the preceding result gives, for $z = 1,\ 2,\ 3,\ \ldots, n$; $\Gamma(2) = 1.\Gamma(1) = 1 = 1!$, $\Gamma(3) = 2.\Gamma(2) = 2.1! = 2!$, $\Gamma(4) = 3.\Gamma(3) = 3.2! = 3!, \ldots$; and, in general,

$$\Gamma(n+1) = n.\Gamma(n) = n.(n-1)! = n!. \tag{4}$$

This justifies the use of these functions as generalization of factorial for the non-integer or complex values of z: $\Gamma(z+1) = z!$. It may be mentioned that $\Gamma(1/2) = \sqrt{\pi}$, and that for n = 0, 1, 2, 3, …, $\frac{1}{\Gamma(-n)} = 0$.

2.2 Exponential Function and Mittag-Leffler Function

A function of great interest in mathematics is the exponential function with base $e = 2.71828\ldots$. For real x, e^x or $\exp(x)$ is given by the power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)}. \tag{5}$$

However, the exponent can be purely imaginary (which gives Euler's relation for cosine and sine functions) and complex z. A straight-forward extension of this function for exponent z reads

$$E_{\rho_1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\rho_1 + 1)}. \tag{6}$$

Here, the additional parameter ρ_1 is real positive. $E_{\rho_1}(z)$ is known as one-parameter or uni-parametric Mittag-Leffler function as it was put forth by him in 1902-03. The properties of this function were meticulously investigated by him as well as Wiman, and later by others. A further extended version defined as

$$E_{\rho_1,\rho_2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(n\rho_1 + \rho_2\right)}, \ (\rho_1,\ \rho_2 > 0), \tag{7}$$

is usually referred to as generalized or two-parameter Mittag-Leffler function though it was introduced and studied by other mathematicians. This function arises naturally in the solution to problems described by differential equations of fractional order. It may be noted that

$$E_{\rho_1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(n\rho_1 + 1\right)} \equiv E_{\rho_1}(z), \tag{8}$$

$$E_{1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = e^z, \tag{9}$$

$$E_{1,2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+2)}$$
$$= \frac{1}{z}\left[-1 + 1 + \sum_{n=0}^{\infty} \frac{z^{n+1}}{\Gamma(n+2)}\right] = \frac{e^z - 1}{z}, \tag{10}$$

$$E_{2,1}\left(-z^2\right) = \sum_{n=0}^{\infty} \frac{\left(-z^2\right)^n}{\Gamma(2n+1)}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos(z), \tag{11}$$

and

$$E_{2,2}\left(-z^2\right) = \sum_{n=0}^{\infty} \frac{\left(-z^2\right)^n}{\Gamma(2n+2)}$$
$$= \frac{1}{z}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \frac{sin(z)}{z}. \tag{12}$$

Furthermore,

$$E_{\rho_1,\rho_1+\rho_2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(n\rho_1 + \rho_1 + \rho_2\right)}$$
$$= \frac{1}{z}\left[\sum_{n=0}^{\infty} \frac{z^{n+1}}{\Gamma\left\{(n+1)\rho_1 + \rho_2\right)\}} + \frac{1}{\Gamma\left(\rho_2\right)} - \frac{1}{\Gamma\left(\rho_2\right)}\right]$$
$$= \frac{1}{z}\left[E_{\rho_1,\rho_2}(z) - \frac{1}{\Gamma\left(\rho_2\right)}\right] \tag{13}$$

and

$$\int_0^z E_{\rho_1,1}\left(az^{\rho_1}\right) dz = \int_0^z \sum_{n=0}^{\infty} \frac{\left(az^{\rho_1}\right)^n}{\Gamma\left(n\rho_1 + 1\right)} \mathrm{d}z$$
$$= \sum_{n=0}^{\infty} \frac{a^n z^{\rho_1 n + 1}}{\left(n\rho_1 + 1\right)\Gamma\left(n\rho_1 + 1\right)} \Big|_0^z = zE_{\rho_1,2}\left(az^{\rho_1}\right). \tag{14}$$

Also, if $0 < \rho_1 < 2$, and ρ_2 is a real arbitrary number, then for limit $z \to \infty$, $E_{\rho_1,\rho_2}(-z) = 0$. It may be mentioned here that in section 5, whenever solution of the fractional differential equation appears as $E_{\rho_1,\,1}(z)$, we shall continue to use this as such rather than replacing it with $E_{\rho_1}(z)$.

3 Basics of Fractional Derivatives

In order to obtain Riemann-Liouville (RL) definition of FD, we consider the n-fold multiple integral

$$_aI^n f(t) =$$
$$\int_a^{t_n = t} \mathrm{d}t_{n-1} \int_a^{t_{n-1}} \mathrm{d}t_{n-2} \ \ldots \ \int_a^{t_1} f\left(t'\right) \mathrm{d}t', \tag{15}$$

where n is a positive integer and f is assumed to be integrable on (a, ∞). Using Cauchy repeated integration formula from calculus together with Eq. (4), we get

$$\begin{aligned} {}_aI^n f(t) &= \frac{1}{(n-1)!}\int_a^t \left(t-t^{'}\right)^{n-1} f\left(t^{'}\right) \mathrm{d}t^{'} \\ &= \frac{1}{\Gamma(n)}\int_a^t \left(t-t^{'}\right)^{n-1} f\left(t^{'}\right) \mathrm{d}t^{'}. \quad (16) \end{aligned}$$

Generalizing this integral of f to fractional order α, we have for $-\infty \le a < t < \infty$,

$$\begin{aligned} {}_aI^\alpha f(t) &= {}_aD^{-\alpha} f(t) \\ &= \frac{1}{\Gamma(\alpha)}\int_a^t \left(t-t^{'}\right)^{\alpha-1} f\left(t^{'}\right) \mathrm{d}t^{'} \\ &= \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f\left(t^{'}\right)}{\left(t-t^{'}\right)^{1-\alpha}} \mathrm{d}t^{'}. \quad (17) \end{aligned}$$

Here, ${}_aD^{-\alpha}$ signifies the fact that it is antiderivative of order α. Such an integral with kernel $\left(t-t^{'}\right)^{\alpha-1}$ is said to define convolution of function f and power of time. Similarly, for $-\infty < t < b \le \infty$,

$$\begin{aligned} {}_bI^\alpha f(t) &= {}_bD^{-\alpha} f(t) \\ &= \frac{1}{\Gamma(\alpha)}\int_t^b \left(t^{'}-t\right)^{\alpha-1} f\left(t^{'}\right) \mathrm{d}t^{'}. \quad (18) \end{aligned}$$

It may be noted that the parameter α can be complex, but we are taking this to be real fractional keeping in mind the applications to be discussed in section 5. Furthermore, the integral in Eq. (17) is valid for $a < t$ and it gets contribution with weight $\left(t-t^{'}\right)^{\alpha-1}$ from $t^{'} < t$, which in the language of geometry means 'from the values to the left of $t^{'}$. So ${}_aI^\alpha f(t)$ is usually referred to as left-handed RL fractional integral of order α. In contrast, the integral defined by Eq. (18) is determined by the contributions for $t^{'} > t$ but less than b and the weight in this case is $\left(t^{'}-t\right)^{\alpha-1}$. Thus, the integral ${}_bI^\alpha f(t)$ exists for values of $t^{'}$ on the right of t so that this is called right-handed RL integral of fractional order α. a and b are said to provide lower and upper bounds or terminal points of the integral domain and can be chosen arbitrarily; these even may be $-\infty$ and ∞, respectively. These statements regarding nonlocality of the fractional integrals are true whatever be the nature of independent variable. However, if the independent variable is time (as we shall use in this article) and t is the instant of observation, then $t^{'} < t$ and $t^{'} > t$ correspond to past and future, respectively. Accordingly, ${}_aI^\alpha f(t)$ is collection of weighted values in the past (from initial time defined by the value of a) and is causal. On the other hand, ${}_bI^\alpha f(t)$ pertains to the future and is anti-causal. In view of this observation, now-on-wards we shall focus our attention only on the left-handed RL integral with lower and upper terminal points as a and t, respectively.

Next, we consider the FD operator ${}_aD^\alpha$ and decompose this as

$$\begin{aligned} {}_aD^\alpha = {}_aD^n{}_aD^{-n+\alpha} = {}_aD^n{}_aI^{n-\alpha} \\ n \in \mathbb{N}, \quad \text{and} \quad n-1 < \alpha \le n. \quad (19) \end{aligned}$$

Thus, we have

$$ {}_aD^{\alpha} f(t) = {}_aD^n {}_aI^{n-\alpha} f(t) $$
$$ = \frac{d^n}{dt^n}[\frac{1}{\Gamma(n-\alpha)}\int_a^t \left(t-t^{'}\right)^{(n-\alpha)-1} f\left(t^{'}\right) \mathrm{d}t^{'}]. \quad (20) $$

This defines the left RL FD ${}^{\mathrm{RL}}_{\ a}D^{\alpha}$ of order α, and, obviously, its operation is equivalent to (left-handed) RL fractional integration of order $(n-\alpha)$ followed by ordinary differentiation of order n. Note that the presence of integral in the expression makes FD to be nonlocal in nature, and that for $\alpha = n-1$, the weight in the integral becomes unity so that the expression on the right ultimately becomes derivative of integer order $n-1$. Generally, Eq. (20) is said to give Liouville FD when a is taken as $-\infty$, and Riemann FD for $a = 0$.

As illustrative examples of Eq. (20), we consider (i) $f(t) = Constant\ (C)$ and (ii) $f(t) = t$, for $n = 1$ so that $0 < \alpha \leq 1$. Substituting these into this equation, we get

(i)

$$ {}^{\mathrm{RL}}_{\ a}D^{\alpha} C = \frac{1}{\Gamma(1-\alpha)} \times \frac{d}{\mathrm{d}t}\left[\int_a^t \left(t-t^{'}\right)^{-\alpha} C\mathrm{d}t^{'}\right] = \frac{C}{\Gamma(1-\alpha)}\frac{d}{\mathrm{d}t}\left[\frac{(t-a)^{1-\alpha}}{(1-\alpha)}\right] = \frac{C(t-a)^{-\alpha}}{\Gamma(1-\alpha)}; \quad (21) $$

(ii)

$$ {}^{\mathrm{RL}}_{\ a}D^{\alpha} t = \frac{1}{\Gamma(1-\alpha)}\frac{d}{\mathrm{d}t}[\int_a^t \left(t-t^{'}\right)^{-\alpha} t^{'}\mathrm{d}t^{'}] $$
$$ = \frac{1}{\Gamma(1-\alpha)}\frac{d}{\mathrm{d}t}\left[t^{'}\frac{-\left(t-t^{'}\right)^{1-\alpha}}{(1-\alpha)} \Big|_{t^{'}=a}^{t} + \int_a^t \frac{\left(t-t^{'}\right)^{1-\alpha}}{(1-\alpha)}\mathrm{d}t^{'}\right] $$
$$ = \frac{1}{\Gamma(1-\alpha)}\frac{d}{\mathrm{d}t}\left[\frac{a(t-a)^{1-\alpha}}{(1-\alpha)} + \frac{(t-a)^{2-\alpha}}{(1-\alpha)(2-\alpha)}\right] $$
$$ = \frac{a(t-a)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)}. \quad (22) $$

For the commonly used lower terminal value of time or initial value in physical problems $a = 0$,

$$ {}^{\mathrm{RL}}_{\ 0}D^{\alpha} C = \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)} \text{ and } {}^{\mathrm{RL}}_{\ 0}D^{\alpha} t = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}. \quad (23) $$

The first result, viz., 'FD of a constant is nonzero and dependent on t', is quite surprising and not acceptable. The second result for $\alpha = \frac{1}{2}$, gives ${}^{\mathrm{RL}}_{\ 0}D^{1/2} t = \frac{t^{1/2}}{\Gamma(3/2)} = 2\sqrt{t/\pi}$, which is the same result as found by Lacroix. Despite the preceding drawback, the RL definition (Eq. (20)) played an important role in the development of FC and its applications in pure mathematics.

To circumvent the problem of nonzero value of FD of a constant, we split ${}_aD^{\alpha}$ into equally acceptable alternative form than that given in Eq. (19) as

$$ {}_aD^{\alpha} = {}_aD^{-n+\alpha} {}_aD^n = {}_aI^{n-\alpha} {}_aD^n \quad n \in \mathbb{N}, \quad \text{and} \quad n-1 < \alpha \leq n. \quad (24) $$

In general, the outcome of the operation defined on the right here is different from that in the previous case, and gives us

$$ {}_aD^{\alpha}f(t) = {}_aI^{n-\alpha}{}_aD^{n}f(t) $$
$$ = \frac{1}{\Gamma(n-\alpha)}[\int_a^t \left(t-t^{'}\right)^{(n-\alpha)-1} \frac{d^n f\left(t^{'}\right)}{dt^{'n}}\mathrm{dt}^{'}]. \quad (25) $$

Here, the determination of FD of order α comprises ordinary differentiation of integer order n and subsequent (left-handed) RL fractional integration of order $(n-\alpha)$. Like the previous case, this derivative is also nonlocal and reduces to derivative of integer order $n-1$ for $\alpha = n-1$. The result obtained from Eq. (25) is known as Caputo derivative ${}_a^CD^{\alpha}$ of order α. Obviously, it is convolution of nth-order derivative of f and a power of time.

As a special case, for $n = 1$ implying $0 < \alpha \leq 1$,

$$ {}_a^CD^{\alpha}C $$
$$ = \frac{1}{\Gamma(1-\alpha)}\frac{d}{\mathrm{dt}}[\int_a^t \left(t-t^{'}\right)^{-\alpha} \frac{\mathrm{dC}}{dt^{'}}\mathrm{dt}^{'}] = 0; \quad (26) $$

and

$$ {}_a^CD^{\alpha}t = \frac{1}{\Gamma(1-\alpha)}[\int_a^t \left(t-t^{'}\right)^{-\alpha} \frac{dt^{'}}{dt^{'}}\mathrm{dt}^{'}] $$
$$ = \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)}. \quad (27) $$

Thus, Caputo FD of a constant vanishes, as it should, and its result for $f(t) = t$, for $a = 0$ is the same as found in Eq. (23). The first result makes Caputo FD formula, Eq. (25), more suitable for practical applications as compared to the RL formula, Eq. (20).

It must be emphasized that the Caputo as well as RL definitions involve differentiation and integration. Therefore, these are differintegrals of fractional order, and the process of determining these derivatives is essentially fractional integrodifferentiation. The presence of power law term $\left(t-t^{'}\right)^{(n-\alpha)-1}$ in the integrals involved in the definition of these two FDs takes care of the memory of the system. Thus, time-evolution of physical systems with memory effects of all types is described in terms of FDs and the analytical solution for the fractional differential equations so obtained is found using Laplace transforms and some other methods. Since the former constitute the basis of the most commonly used approach, we dwell upon this topic in the next section.

However, before proceeding in this direction, it is worth mentioning that RL FD based differential equations require the initial conditions to be expressed in terms of initial values of the FDs of yet-to-be-determined function $f(t)$, which do not have any direct connection with experimental findings; of course, some way outs have been put forth. Nonetheless, this limitation together with the observation made after Eq. (23) leads to a serious constraint on the use of RL formulation in practical problems. On the other hand, the initial conditions involved in the fractional differential equations with Caputo derivative correspond to physically definable integer-order derivatives of $f(t)$. Furthermore, as shown in Eq. (26), ${}^C_aD^\alpha C = 0$. Accordingly, this formula is usually preferred for handling problems in physics and engineering, and this will be the one to be used in the remaining part of the article. For convenience, we shall drop C as well as a (taken to be 0 as the initial time) from the symbol ${}^C_aD^\alpha$for the relevant operator and denote it simply by D^α. It may be pointed out that the value of the order of derivative provides a measure of the memory of the system - a lower value implies higher memory effect and vice versa.

Furthermore, by carrying out integration by parts repeatedly in Eq. (25), it is found that for $n = 1$ (implying $0 < \alpha \leq 1$),

$$D^\alpha t^j = \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}\, t^{j-\alpha}, j > -1. \quad (28)$$

It is interesting to note that n repeated differentiations of t^j, for $n \leq j$, give

$$D^n t^j = \frac{d^n t^j}{dt^n} = \frac{j!}{(j-n)!} t^{j-n} = \frac{\Gamma(j+1)}{\Gamma(j+1-n)}\, t^{j-n}. \quad (29)$$

A look at these two results shows that the former can be taken as an extension of the latter for a FD. A similar generalization of Eq. (28) yields expression for FD of fractional power of t.

4 Laplace Transforms

The Laplace transform (LT) is an integral transform that converts a function $f(t)$ of real variable t (generally time) into a function $F(s)$ of complex variable s (complex frequency) defined by

$$F(s) \equiv \mathcal{L}\left\{f(t)\right\}(s) = \int_0^\infty e^{-st} f(t)\mathrm{dt}. \quad (30)$$

This integral exists if $f(t)$ is piecewise continuous for $0 \leq t < \infty$, and of exponential order ζ meaning that there exist a constant ζ, and positive constants M and t_0 such that $|f(t)|\, e^{-\zeta t} \leq M$ for all $s > \zeta$ and $t \geq t_0$. The weight e^{-st} in Eq. (30) is called kernel of the transform operator $\mathcal{L}$. The transform so defined is usually referred to as unilateral or one-sided, while the one with lower limit of integration as $-\infty$ is said to be bilateral. However, we shall be using only the former and shall ignore the adjective unilateral. As we proceed, we shall see that this transform converts a (fractional) differential equation for $f(t)$ in time domain into an algebraic equation in $F(s)$ in the complex frequency domain, which is much easier to handle than actually solving the differential equation. The solution so obtained is then subjected to inverse Laplace transformation to yield the solution for the differential equation, $f(t) = \mathcal{L}^{-1}F(s)$, with the given initial conditions. The rigorous procedure for evaluation of inverse Laplace transform from this equation is quite complicated, and, generally, the tables listing $f(t)$ and corresponding $F(s)$ are used for this purpose.

The LT for given function $f(t)$ is obtained from Eq. (30) by using integration by parts. The results for some functions of our interest are given below.

$$\mathcal{L}\left\{C\right\} = \frac{C}{s}\,,$$
$$\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}\,,$$
$$\mathcal{L}\left\{t^\alpha\right\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}},$$
$$\mathcal{L}\left\{e^{\pm\sigma t}\right\} = \frac{1}{s\,\mp\,\sigma}; \qquad (31)$$

and

$$\mathcal{L}\left\{t^{\rho_1 l+\rho_2-1}E^{(l)}_{\rho_1,\rho_2}(\pm a t^{\rho_1})\right\} = \frac{l!\;s^{\rho_1-\rho_2}}{(s^{\rho_1}\mp a)^{l+1}},$$
$$\mathrm{Re}(s) > |a|^{1/\rho_1}, \qquad (32)$$

where $E^{(j)}_{\rho_1,\rho_2}(y) = \frac{d^j}{\mathrm{d}y^j}E_{\rho_1,\rho_2}(y)$. Furthermore, the LT of the Caputo FD of order α, $(n-1 < \alpha \;\leq n)$, is given by

$$\mathcal{L}\left\{D^\alpha f(t)\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0). \qquad (33)$$

The presence of $f^{(k)}(0)$, viz., the values of the function $f(t)$ and its derivatives of integer-order at $t = 0$, in this expression establishes direct contact with the initial conditions of the physical system, which makes these useful for practical applications. Note that Eq. (33) reduces to LT of integer-order derivatives for $\alpha = n$.

5 Some Illustrative Examples from Elementary Physics

Having equipped ourselves with the essentials of FC, we now use this to discuss some simple problems from introductory physics in this section.

5.1 Newton's Second Law and One-Dimensional Equations of Motion

One of the most important laws of physics which finds wide range applications in almost all branches of science and engineering, the Newton's second law of motion states that for a material point or particle of constant mass m, the time rate of change of its momentum (mass $\times$ velocity) equals the magnitude of the force applied and both have the same direction. Thus, for a one-dimensional motion with instantaneous velocity $v(t)$ under the influence of constant force f, it reads

$$m\frac{dv(t)}{\mathrm{d}t} = f. \qquad (34)$$

Replacing the ordinary first-order derivative by a Caputo FD of arbitrary order α $(0 < \alpha \;\leq 1)$, this can be written as

$$m_F\frac{d^\alpha v(t)}{dt^\alpha} \equiv m_F D^\alpha v(t) = f. \qquad (35)$$

Obviously, Eq. (35) is fractional or generalized form of the Newton's second law of motion. To keep the meaning of $v(t)$ and f same as in Eq. (34), m_F must have dimension $MT^{\alpha-1}$. Mass parameter m_F becomes mass m for index value $\alpha = 1$. Using Eq. (33) with $n = 1$ and first expression in Eq. (31), we get the LT of the two sides of Eq. (35) as

$$m_F\left[s^\alpha V(s) - s^{\alpha-1}v(0)\right] = \frac{f}{s}, \qquad (36)$$

with $v(0)$ as initial velocity. Note that the fractional differential equation for $v(t)$, viz.

Eq. (35), has been converted into an algebraic equation in $V(s)$. This gives us

$$V(s) = \frac{f}{m_F}\frac{1}{s^{\alpha+1}} + \frac{v(0)}{s}. \quad (37)$$

Inverse LT of this equation yields

$$v(t) = v(0) + \frac{f}{m_F}\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \quad (38)$$

where we have used the third relation in Eq. (31). This is equation of motion for velocity in the framework of FC, and for $\alpha = 1$, it reduces to the well-known equation of motion,

$$v(t) = v(0) + \frac{f}{m}t = v(0) + at, \quad (39)$$

with $a = \frac{f}{m}$ as constant acceleration. The dependence of $v(t)$ on t for different values of α for $v(0)$ =1.0 ms^{-1}, f= 1 N, and m_F= 1 kgs$^{\alpha-1}$ is depicted in Fig. 1. As expected from Eq. (38), for $t > 1$ s variation in velocity $v(t)$ with time is slow for small values of α, as if the motion is facing some time-delay causing dissipative resistance. In fact, this is a manifestation of memory effects implying their higher relevance for smaller α. Note that these effects become more prominent for higher values of t.

Next, in view of the fact that $v(t) = \frac{dx(t)}{\mathrm{dt}}$, where $x(t)$ is instantaneous position of the particle, Eq. (34) can be written as $m\frac{d^2x(t)}{\mathrm{dt}^2} = f$, and the corresponding differential equation for arbitrary order β ($1 < \beta \leq 2$), reads

$$m_F^{'}\frac{d^{\beta}x(t)}{dt^{\beta}} \equiv m_F^{'}D^{\beta}x(t) = f, \quad (40)$$

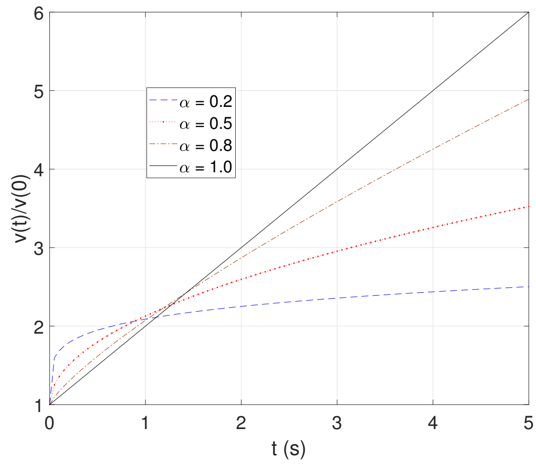

Figure 1: Plots showing velocity as function of time for different values of fractional order derivative parameter α for $v(0) = 1.0$ ms^{-1}, f = 1 N, and m_F= 1 kgs$^{\alpha-1}$.

Here, the mass parameter $m_F^{'}$ has dimension $MT^{\beta-2}$, and for $\beta = 2$, $m_F^{'} = m$. Finding LT of both sides, with Eq. (33) for $n = 2$, and rearranging various terms, we have

$$X(s) = \frac{f}{m_F^{'}}\frac{1}{s^{\beta+1}} + \frac{x^{(1)}(0)}{s^2} + \frac{x(0)}{s}. \quad (41)$$

The inverse LT of this equation, together with the fact that $x^{(1)}(0) = v(0)$, gives us

$$x(t) = x(0) + v(0)t + \frac{f}{m_F^{'}}\frac{t^{\beta}}{\Gamma(\beta+1)}, \quad (42)$$

which is equation of motion for position at time t. For the special value $\beta = 2$, this reads

$$x(t) = x(0) + v(0)t + \frac{1}{2}\,\mathrm{a}t^2, \quad (43)$$

the so called second equation of motion. The graphical representation for variation of $x(t)$ as function of t for $x(0) = 1.0$ m, $v(0) = 1.0$ ms^{-1}, f = 1 N, and $m_F^{'}$= 1 kgs$^{\beta-2}$ for different

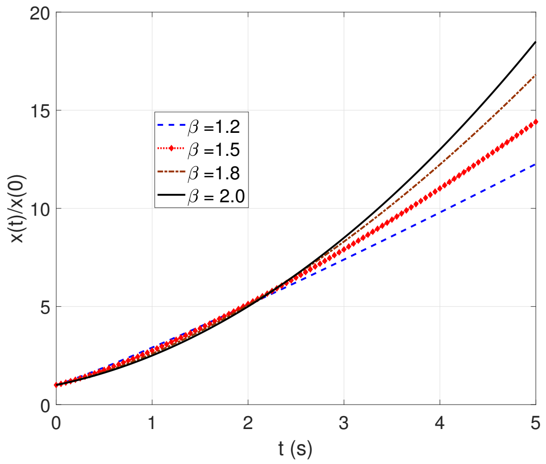

Figure 2: Variation of position of particle with time for different values of fractional order derivative parameter β. The parameters used are $x(0) = 1.0$ m, $v(0) = 1.0$ ms^{-1}, f = 1 N, and m'_F= 1 kgs$^{\beta-2}$.

values of β is projected in Fig. 2. Once again, the departure of graphs for $1 < \beta < 2$, from the conventional case of $\beta = 2$ is a consequence of memory effects. And this becomes more noticeable for t > 2 s.

It is worth mentioning that in the framework of FC, linear momentum at any time t is expressed in terms of position at that time by

$$p(t) = m'_F \frac{d^{\beta/2}x(t)}{dt^{\beta/2}}, \quad 1 < \beta \leq 2. \qquad (44)$$

Substituting for $x(t)$ from Eq. (42) and using the fact that the order of derivative $\frac{\beta}{2}$ satisfies the condition for Eq. (28), we get, using this equation and its generalization for non-integer power of t,

$$p(t) = m'_F [\frac{\Gamma(2)}{\Gamma(2-\frac{\beta}{2})} v(0)\, t^{1-\frac{\beta}{2}} + \frac{f}{m'_F} \frac{t^{\frac{\beta}{2}}}{\Gamma\left(1+\frac{\beta}{2}\right)}]. \qquad (45)$$

For $\beta = 2$, this becomes

$$p(t) = m\left[v(0) + \frac{f}{m}\,\mathrm{t}\right] = mv(t), \qquad (46)$$

which is the conventional expression for instantaneous momentum. Also, for this value of β, Eq. (44) reads $p(t) = m\frac{dx(t)}{\mathrm{dt}}$. These observations justify the above definition of $p(t)$.

5.2 An Object Falling Under Gravity in a Medium Offering Resistance

When a particle-like small object (say, a solid metallic ball) falls from an initial height $y(0)$ under the influence of a constant gravitational field through a viscous fluid (castor oil, glycerine, sugar syrup, etc), or a large body (say, a human being in swimming pose) falls through a low-viscosity fluid (water, or even air), it experiences a viscosity-associated upward resistive or drag force that opposes its downward motion with respect to the surrounding medium. The magnitude of this drag force is proportional to the magnitude of instantaneous relative velocity $v(t)$ for slow speeds and to its higher powers for high speeds. Taking upward motion to be positive and assuming the speed to be low, the equation of motion for a particle of mass m is given by

$$-m\frac{\mathrm{dv}(t)}{\mathrm{dt}} = -mg + \gamma v(t). \qquad (47)$$

Here, γ is the linear drag or friction coefficient, which is determined by the shape, material, and size of the falling body, and by the nature and temperature of the fluid. Using Caputo fractional derivative of order α on the left side, we get the corresponding fractional equation as

$$m_F D^\alpha v(t) = mg - \gamma v(t), \quad 0 < \alpha \leq 1. \tag{48}$$

Taking LT on both sides and rearranging the terms, we get

$$V(s) = \frac{\text{mg}}{m_F} \frac{s^{-1}}{(s^\alpha + \frac{\gamma}{m_F})} + v(0) \frac{s^{\alpha-1}}{(s^\alpha + \frac{\gamma}{m_F})}. \tag{49}$$

Finding inverse LT, in the light of Eq. (32), we have

$$\begin{aligned} v(t) &= \frac{\text{mg}}{m_F} t^\alpha E_{\alpha,\, \alpha+1}(-\frac{\gamma}{m_F} t^\alpha) \\ &\quad + v(0) E_{\alpha,1}(-\frac{\gamma}{m_F} t^\alpha) \\ &= \frac{\text{mg}}{m_F} t^\alpha \frac{1}{-\frac{\gamma}{m_F} t^\alpha} \left[E_{\alpha,\,1}\left(-\frac{\gamma}{m_F} t^\alpha\right) - 1 \right] \\ &\quad + v(0) E_{\alpha,1}(-\frac{\gamma}{m_F} t^\alpha) \\ &= \frac{\text{mg}}{\gamma} + [v(0) - \frac{\text{mg}}{\gamma}] E_{\alpha,1}(-\frac{\gamma}{m_F} t^\alpha). \end{aligned} \tag{50}$$

Here, we have used Eq. (13) to go from the first equality to the second one. Denoting the instantaneous position of the body by $y(t)$, we have $v(t) = \frac{dy(t)}{\text{dt}}$, and, hence

$$\begin{aligned} y(t) - y(0) &= \int_0^t v(t) \text{dt} \\ &= \frac{\text{mg}}{\gamma} t + [v(0) - \frac{\text{mg}}{\gamma}] \int_0^t E_{\alpha,1}(-\frac{\gamma}{m_F} t^\alpha) dt. \end{aligned} \tag{51}$$

Substituting for the integral from Eq. (14), we have

$$y(t) = y(0) + \frac{\text{mg}}{\gamma} t + [v(0) - \frac{\text{mg}}{\gamma}] t E_{\alpha,2}(-\frac{\gamma}{m_F} t^\alpha). \tag{52}$$

It may be remarked that the Mittag-Leffler functions appearing in Eqs. (50) and (52) are monotone so that the expressions for $v(t)$ as well as $y(t)$ are physically acceptable.

In view of the statement after Eq. (14), the Mittag-Leffler function in Eq. (50) vanishes for infinite values of t, and, therefore,

$$v(t)|_{t\to\infty} = \frac{\text{mg}}{\gamma}, \tag{53}$$

irrespective of the value of $v(0)$. This gives the terminal speed of the particle, and, interestingly, it depends upon m and not m_F. Note that if $v(0)$ equals terminal velocity, then the term containing $E_{\alpha,1}(-\frac{\gamma}{m_F} t^\alpha)$ in Eq. (50) vanishes and $v(t) = \frac{\text{mg}}{\gamma}$ for all values of t irrespective of the value of α.

For $\alpha = 1$, Eqs. (50) and (52), together with Eqs (9) and (10) give

$$v(t) = \frac{\text{mg}}{\gamma} + [v(0) - \frac{\text{mg}}{\gamma}] e^{-\frac{\gamma}{m} t}, \tag{54}$$

and

$$\begin{aligned} y(t) &= y(0) + \frac{\text{mg}}{\gamma} t + \left[v(0) - \frac{\text{mg}}{\gamma} \right] t \frac{e^{-\frac{\gamma}{m} t} - 1}{-\frac{\gamma}{m} t} \\ &= y(0) + \frac{\text{m}}{\gamma} [gt - \{v(0) - \frac{\text{mg}}{\gamma}\}(e^{-\frac{\gamma}{m} t} - 1)]. \end{aligned} \tag{55}$$

These are the same results as we obtain by solving the differential equation in Eq. (47). Furthermore, the terminal speed of the particle as determined from Eq. (54) is the same as found in Eq. (53).

In order to explore the effect of fractional order parameter α on the velocity of a falling body, we have used Eq. (50) to obtain plots for $v(t)$ as function of time using $m = 0.02$ kg, $m_F = 0.02$ kgs$^{\alpha-1}$, v_0 = 0 (the object is just dropped), g = 9.81 ms^{-2}, γ = 0.05 kgs^{-1} ; Fig. 3. It may be mentioned that the terminal velocity found through Eq. (53) is 3.924 ms^{-1}. For $\alpha = 1.0$, this value is attained at $t = 3.4$ s, whereas for lower values of α, this does not happen even at $t = 8$ s, for which the $v(t)$ values are 3.816, 3,852, and 3.896 ms^{-1} for $\alpha = 0.7$, 0.8, and 0.9, respectively. This slowness or time-delay in reaching the terminal velocity is manifestation of memory effects and is more pronounced for lower α.

It is worth pointing out that if $v_0 \left(< \frac{\mathrm{mg}}{\gamma}\right)$ is taken to be nonzero, then not only ordinates of the plots start from this value but even the asymptotic values for higher t values become closer to the terminal velocity. For example, if we take $v_0 = 1$ ms^{-1} and 3 ms^{-1}, then the values of $v(t)$ at $t = 8$ s for $\alpha = 0.7$ are found to be 3.844 and 3.899 ms^{-1}, respectively, in lieu of 3.816 ms^{-1} for v_0 = 0. Furthermore, if $v_0 > \frac{\mathrm{mg}}{\gamma}$, then the falling body decelerates to attain the terminal velocity from above and the dependence of time variation of velocity on α appears as in Fig. 4, where v_0 = 6 ms^{-1}. In this case, $v(t)$ at $t = 8$ s for $\alpha = 0.7$ and 1.0 has values 3.981 and 3.924 ms^{-1}, respectively.

In contrast with the above, if the falling body is quite small and the surrounding medium has fairly low viscosity, say air whose coefficient of viscosity is nearly 80,000 times less than that of glycerine, the drag force can be taken to be negligible as compared to the gravitational force or weight. The formulae describing the motion of the body falling under this friction-free ideal condition can be obtained from Eqs. (50) and (52) by taking the limit $\gamma \to 0$. For this purpose, we first write $E_{\alpha,1}(-\frac{\gamma}{m_F}t^\alpha)$ and $E_{\alpha,2}(-\frac{\gamma}{m_F}t^\alpha)$ in these two equations as sum using Eq. (7), simplify the expressions and take the limit. This gives us, for the so called freely falling body,

$$v(t) = v(0) + \frac{\mathrm{mg}}{m_F}\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{56}$$

and

$$y(t) = y(0) + v(0)t + \frac{\mathrm{mg}}{m_F}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}; \tag{57}$$

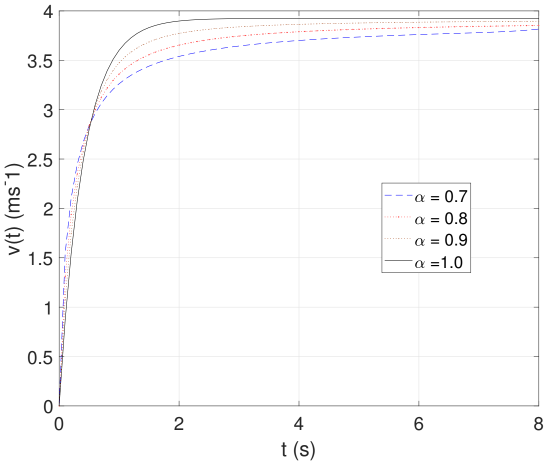

Figure 3: Time-dependence of velocity of a falling body for different values of α, The parameters used for obtaining these plots are $m = 0.02$ kg, $m_F = 0.02$ kgs$^{\alpha-1}$, v_0 = 0 ms^{-1}, g = 9.81 ms^{-2}, γ = 0.05 kgs^{-1}.

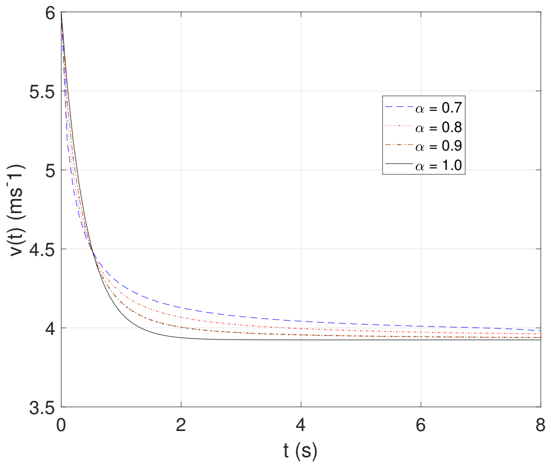

Figure 4: Velocity of a falling body as function of t for different values of α and v_0 = 6.0 ms^{-1}. Other parameters are the same as used in Fig. 3.

which for $\alpha = 1$, become $v(t) = v(0) + gt$, and $y(t) = y(0) + v(0)t + \frac{1}{2}gt^2$. It can be verified that the same results are obtained if we put $\gamma = 0$ in Eqs. (47) and (48), yielding $m\frac{\mathrm{dv}(t)}{\mathrm{dt}} = mg$ and $m_F D^\alpha v(t) = mg$, respectively, and solving these adopting the procedure followed for Eq. (48).

A perusal of Eq. (56) shows that in the absence of resistive force, velocity of the body continuously increases at a rate governed by the value of α , and will never attain a terminal speed which is characteristic of drag.

It may be mentioned that if the object under consideration has a charge q and is subjected to a downward (or upward) acting constant electric field of magnitude $\mathcal{E}$, it will experience a downward (or upward) force $q\mathcal{E}$, and mg will be replaced by $mg + q\mathcal{E}$ (or $mg - q\mathcal{E}$) in all the preceding expressions.

5.3 Projectile Motion in a Viscous Medium

A projectile signifies a particle or a body thrown or projected with some initial velocity into a medium such that it moves along a curved path in a vertical plane under the influence of downward acting gravitational force and no other propelling force. Depending on the nature of the projected body and the surrounding medium, it may or may not experience viscous friction force opposing its horizontal as well as vertical motion. Some typical examples of projectiles are a shell fired by a tank, any ball hit in the air by a sports person, an arrow shot by an archer, etc.

Suppose a projectile of mass m is thrown from point (x_0, y_0) with velocity $v(0) = v_0$ at angle of elevation θ with positive x – direction so that $v_x(0) = v_0 \cos\theta$ and $v_y(0) = v_0 \sin\theta$. We assume that the heights involved are such that the body is always under the influence of the same gravitational field. Taking the upward motion to be positive and assuming that the drag force is linearly dependent on velocity, the conventional equations describing the motion of the projectile in a resistive fluid with friction coefficient γ can be written as

$$m\frac{dv_x(t)}{\mathrm{dt}} = -\gamma v_x(t), \qquad (58a)$$

$$\mathrm{m}\frac{dv_y(t)}{\mathrm{dt}} = -mg - \gamma v_y(t). \qquad (58b)$$

The corresponding fractional differential equations are

$$m_F D^\alpha v_x(t) = -\gamma v_x(t), \qquad (59a)$$

$$m_F D^\alpha v_y(t) = -mg - \gamma v_y(t) \quad (0 < \alpha \ \leq 1). \tag{59b}$$

Finding LT for both sides of Eqs. (59), rearranging the terms to get $V_x(s)$ and $V_y(s)$, and then determining the inverse Laplace transforms, we get

$$v_x(t) = v_0 \ \cos\theta E_{\alpha,1}(-\frac{\gamma}{m_F}t^\alpha), \tag{60}$$

and

$$\begin{aligned} v_y(t) = & -\frac{\text{mg}}{m_F}t^\alpha E_{\alpha,\ \alpha+1}(-\frac{\gamma}{m_F}t^\alpha) \\ & + \ v_0 \ \sin\theta \ E_{\alpha,1}(-\frac{\gamma}{m_F}t^\alpha) \\ = & -\frac{\text{mg}}{\gamma} + [\ v_0 \ \sin\theta + \frac{\text{mg}}{\gamma}]E_{\alpha,1}(-\frac{\gamma}{m_F}t^\alpha). \end{aligned} \tag{61}$$

Integrating both sides in Eqs. (60) and (61) from 0 to t, and using Eq. (14), we have

$$x(t) = x(0) + v_0 \ \cos\theta \ \text{t}E_{\alpha,2}\left(-\frac{\gamma}{m_F}t^\alpha\right), \tag{62}$$

and

$$\begin{aligned} y(t) = \ & y(0) - \frac{\text{mg}}{\gamma}t + [\ v_0 \ \sin\theta \\ & + \frac{\text{mg}}{\gamma}]\ tE_{\alpha,2}(-\frac{\gamma}{m_F}t^\alpha). \end{aligned} \tag{63}$$

For convenience, we take the point of launch of the projectile as origin of the coordinate system so that $x(0) = 0$ and $y(0) = 0$. Then from Eq. (62), we have

$$t = \frac{x(t)}{v_0 \ \cos\theta \ \text{E}_{\alpha,2}(-\frac{\gamma}{m_F}t^\alpha)} \tag{64}$$

Substituting this into Eq. (63) and simplifying the resulting expression, we get

$$y = \ x\tan\theta + \frac{\text{mg}}{\gamma v_0 \ \cos\theta}x[\ 1 - \frac{1}{E_{\alpha,2}(-\frac{\gamma}{m_F}t^\alpha)}]. \tag{65}$$

This defines trajectory of the projectile. The value of $x(t)$ for which $y(t)$ again equals $y(0)$, gives range of the projectile and time needed to cover this distance, as obtained from Eq. (64), is the time of flight. Obviously, these and other characteristics of the motion of the projectile depend on the values of θ and α besides its mass, initial velocity, and friction coefficient. Once again, $\alpha = 1$ leads to results for the motion of a projectile described by Eq. (58).

Next, if the resisting drag of the medium is negligibly small, we can determine the corresponding results by evaluating Eqs. (62) and (63) in the limit $\gamma \to 0$ as done in subsection 5.2. This gives us

$$x(t) = x(0) + v_0 \ t\cos\theta, \tag{66}$$

and

$$y(t) = \ y(0) + v_0 \ t\sin\theta - \frac{\text{mg}}{m_F}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{67}$$

Taking $x(0) = y(0) = 0$, so that $t = \frac{x(t)}{v_0}\sec\theta$, we have for the trajectory of a drag-free projectile,

$$y = \ x\tan\theta - \frac{\text{mg}}{m_F}\frac{(\frac{x}{v_0}\sec\theta)^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{68}$$

For $\alpha = 1$, the usual (inverted parabola) expression for trajectory of a projectile, viz.,

$$y = \ x\tan\theta - \frac{gx^2 \ \sec^2\theta}{2{v_0}^2}, \tag{69}$$

is regained.

As a typical illustration of Eq. (65), we have shown the dependence of the trajectory of a projectile with $m_F = 1$ kgs$^{\alpha-1}$,

v_0 = 12 ms^{-1}, g = 9.81 ms^{-2}, γ = 0.05 kgs^{-1}, $\theta = \pi/4$ for different values of α in Fig. 5. The ranges for α = 0.5, 0.7, 0.9, and 1.0 are found to be 10.65, 12.16, 13.34, and 13.90 m, respectively. It may be noted that for the classical integer order case with γ= 0, the range is $v_0{}^2 sin(2\theta)/g = 14.68$ m (maximum because $\theta = \pi/4$), which is higher by 0.78 m as compared to the range in the presence of drag force characterized by $\frac{\gamma}{m_F}$ = 0.05 $s^{-\alpha}$.

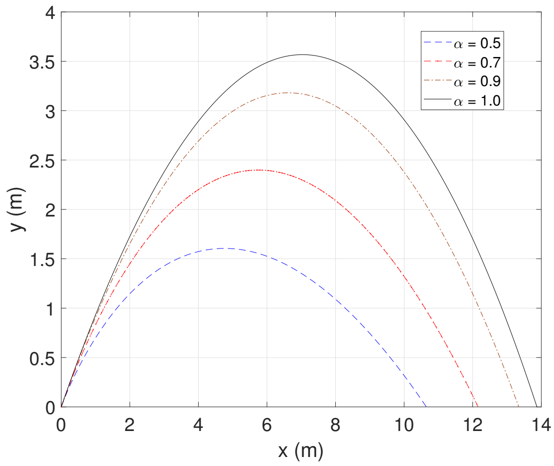

Figure 5: Dependence of trajectory of a projectile ($m_F = 1$ kgs$^{\alpha-1}$, v_0 = 12 ms^{-1}, g = 9.81 ms^{-2}, γ = 0.05 kgs^{-1}, $\theta = \pi/4$) on α.

5.4 Fractional Oscillator

A particle or an object executing repeated back and forth motion after being displaced from its equilibrium position in such a way that the restoring force is always directed towards the point of equilibrium and its magnitude is proportional to the displacement, is called a mechanical harmonic oscillator (HO). In one dimension, for a body of mass m having instantaneous displacement $x(t)$ with equilibrium or centre point at $x = 0$, such a motion (known as simple harmonic motion) is described by the differential equation,

$$m\frac{d^2x(t)}{dt^2} = -kx(t). \tag{70}$$

Here, the negative sign takes care of the fact that the directions of force and the displacement are opposite to each other. The parameter k, the force per unit displacement, is called force constant. In case, the restoring force depends on higher powers of displacement, then the oscillator is said to be anharmonic. If the oscillating system experiences a friction-like force (which is generally always present in real systems) that causes continuous dissipation of energy, then it is referred to as a damped oscillator. Furthermore, an oscillator subjected to a time dependent external force is known as a forced or driven oscillator. The basic equation (70) is accordingly amended to accommodate relevant additional terms.

In general, any system, which need not necessarily be a material object and may be something like electric or magnetic field, that can be described by an expression analogous to that in Eq. (70) or its modified version, is said to be oscillatory. Besides everyday life examples of swings, cradles, vehicle shock absorbers, musical instruments, process of hearing, etc., considered in its general sense, an oscillator (classical or quantum one- or three- dimensional and their further modifications) is ubiquitous in physics and finds wide-range applications in developing theoretical models for different phenomena in almost all branches of physics. Some typical examples are simple, compound, and torsional pendulums; spring-mass system; vibrating tuning forks, strings, and gas columns; electric LC and LCR circuits; electronic oscillators; vibration of atoms in molecules; vibration of lattice atoms and molecules in solids leading to creation of phonons; modelling nuclear collective motion; basics of quantum field theory; and so on.

In view of Eq. (70), the FD equation for a one-dimensional oscillator can be written as

$$m_F D^{\beta} x(t) = -kx(t), 1 < \beta \leq 2; \qquad (71)$$

where m_F has dimension $MT^{\beta-2}$. The LT of this yields

$$X(s) = \frac{x(0)s^{\beta-1}}{\left(s^{\beta} + \frac{k}{m_F}\right)} + \frac{v(0)s^{\beta-2}}{(s^{\beta} + \frac{k}{m_F})}. \qquad (72)$$

This gives analytical expression for instantaneous displacement as

$$x(t) = x(0)E_{\beta,1}\left(-\frac{k}{m_F}t^{\beta}\right) + v(0)tE_{\beta,2}\left(-\frac{k}{m_F}t^{\beta}\right). \qquad (73)$$

We get only one of these two terms if either $v(0) = 0$ or $x(0) = 0$. Note that the condition $v(0) = 0$, $x(0) \neq 0$ implies that the oscillator is at the turning point at $t = 0$ and $x(0)$ is its amplitude. In contrast, the initial condition $x(0) = 0$, $v(0) \neq 0$ corresponds to the situation that the oscillator starts its motion from the equilibrium point at the origin with velocity $v(0)$.

Next, the total mechanical energy of the fractional oscillator at time t is given by

$$\mathbb{E}(t) = \left(\frac{p^2(t)}{2m_F}\right) + (\frac{\mathrm{k}x^2(t)}{2}).$$

To evaluate this, we use Eq. (44) for $p(t)$, and assume the initial conditions to be such that $v(0) = 0$, so that

$$x(t) = x(0)E_{\beta,1}\left(-\frac{k}{m_F}t^{\beta}\right); \qquad (74)$$

$$\begin{aligned} p(t) &= m_F \frac{d^{\frac{\beta}{2}}\left[x(0)E_{\beta,1}\left(-\frac{k}{m_F}t^{\beta}\right)\right]}{dt^{\frac{\beta}{2}}} \\ &= m_F x(0)\sum_{n=0}^{\infty}\left(-\frac{k}{m_F}\right)^n \frac{t^{\beta\left(n-\frac{1}{2}\right)}}{\Gamma\left(n\beta+1-\frac{\beta}{2}\right)} \\ &= -m_F x(0)\frac{k}{m_F}t^{\beta/2}E_{\beta,1+\beta/2}\left(-\frac{k}{m_F}t^{\beta}\right). \end{aligned} \qquad (75)$$

Therefore,

$$\begin{aligned} \mathbb{E}(t) &= \frac{1}{2}m_F x^2(0)(\frac{k}{m_F})^2 t^{\beta}[E_{\beta,1+\beta/2}\left(-\frac{k}{m_F}t^{\beta}\right)]^2 \\ &+ \frac{1}{2}kx^2(0)[E_{\beta,1}\left(-\frac{k}{m_F}t^{\beta}\right)]^2. \end{aligned} \qquad (76)$$

Obviously, $\mathbb{E}(t)$ varies with time implying that the total mechanical energy of a fractional oscillator is not a constant or is not conserved.

Furthermore, for maximum allowed value of β, i.e., $\beta = 2$, using Eqs. (11) and (12), we get from Eq. (73),

$$x(t) = x(0)\cos(\sqrt{\frac{k}{m}}\,t) + v(0)\sqrt{\frac{m}{k}}\,\sin(\sqrt{\frac{k}{m}}\,t). \qquad (77)$$

This can be easily identified as displacement of a conventional HO with natural angular frequency $\omega = \sqrt{\frac{k}{m}}$. The initial condition $v(0) = 0$ leads to $x(t) = x(0)\cos(\sqrt{\frac{k}{m}}\, t)$. Next, substituting $\beta = 2$ into Eqs. (75) and (76), we get

$$p(t) = -\mathrm{m}\, x(0)\frac{k}{m}tE_{2,2}\left(-\frac{k}{m_F}t^2\right) = -\mathrm{m}\, x(0)\sqrt{\frac{k}{m}}sin(\sqrt{\frac{k}{m}}\,\mathrm{t}), \quad (78)$$

and

$$\mathbb{E}(t) = \frac{1}{2}mx^2(0)\left(\frac{k}{m}\right)^2 t^2\left[E_{2,2}\left(-\frac{k}{m}t^2\right)\right]^2 + \frac{1}{2}kx^2(0)[E_{2,1}\left(-\frac{k}{m}t^2\right)]^2 = \frac{1}{2}\,\mathrm{k}x^2(0). \quad (79)$$

Note that expression in Eq. (78) is the same as is obtained by taking $p(t) = m\frac{dx(t)}{\mathrm{dt}}$. Also, Eq. (79) brings out conservation of mechanical energy for the conventional HO, which is in contrast with the finding in Eq. (76) for a fractional oscillator.

The time dependence of $x(t)$ and $\mathbb{E}(t)$ for an oscillator with $\sqrt{\frac{k}{m_F}}$ = 1.0 rad $s^{-\beta/2}$, as given by Eqs. (74) and (77), respectively, for different β values are shown in Figs. 6 and 7. A look at Fig. 6 shows that for $\beta = 2$, the plot represents a cosine variation so that displacement (of the standard oscillator) is periodic and has same amplitude. However, for $1 < \beta < 2$, the system makes a finite number of oscillations with decreasing amplitudes and finally decays to $x(t) = 0$ position. This attenuation of oscillations is similar to that observed in a damped HO. Moreover, this effect becomes more prominent as β decreases; the number of zeros in the $x(t)$ plots is quite small for β close to 1 (implying quite rapidly damped oscillations). In the case of total mechanical energy (Fig. 7), the graph for $\beta = 2$ corresponds to constancy of $\mathbb{E}(t)$ and, hence, the conservation of energy, while the $\beta < 2$ curves exhibit decrease in energy with time as for the damped or dissipative motion. Once again, the decrease in the curve is faster for lower values of β. Thus, a fractional oscillator ($1 < \beta < 2$) behaves as a damped oscillator even though there is no resistive medium and this damping is absent for $\beta = 2$, which corresponds to a simple HO. In other words, the past-history of the oscillator with $1 < \beta < 2$ influences its motion by producing dampening effect as if it were interacting with itself – memory effect. Accordingly, it is usually referred to as fractional-order intrinsic damping. Since the effect becomes more pronounced with decrease in β, the order of fractional derivative provides a measure of memory effects.

With a view to look at this feature more minutely, we recall that damping and hence dissipation in a standard HO is obtained when a velocity-dependent term ($-\gamma\frac{\mathrm{dx}(t)}{\mathrm{dt}}$) is added to the restoring force term on the right side of Eq. (70) so that this is external in nature. It is indeed interesting to note that though we started with zeroth order (i.e., no) derivative in the restoring force in Eq. (71), we have also got an effect associated with first-order derivative of $x(t)$. This means that the fractional derivative on the

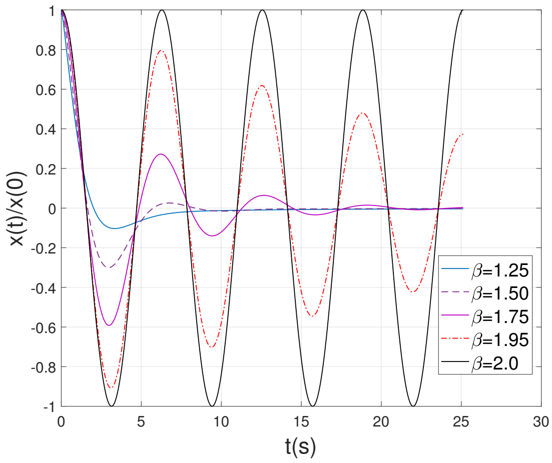

Figure 6: x(t)/x(0) versus time plots for an oscillator ($\sqrt{\frac{k}{m_F}}$ = 1.0 rad $s^{-\beta/2}$) for different β values.

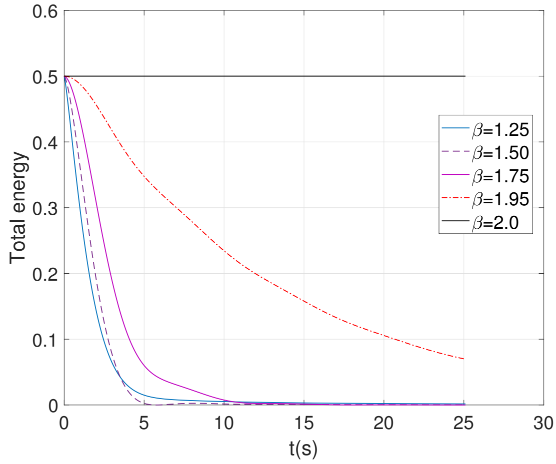

Figure 7: Time-dependence of total mechanical energy of oscillator ($\sqrt{\frac{k}{m_F}}$ = 1.0 rad $s^{-\beta/2}$) for different values of β.

left side of this equation has led to a behaviour which is a mixture of zeroth order and first order derivatives. The inclusion of (ubiquitous) dissipation effect in a natural way in the description makes it closer to reality, and, thus, brings out the importance of FC as a logical and comprehensive tool for modeling real systems by incorporating memory effects.

It is worth pointing out that Rekhviashvili et al [11] studied vibrations of a free piezoelectric plate under standard laboratory conditions and found that their experimental data was best accounted for with $\beta = 1.998$ (reasonably small memory effect).

Before closing this subsection, it may be mentioned that non-conservation of mechanical energy is also observed in the FC description of a freely falling body discussed in subsection 5.2.

5.5 Fourier's Law of Thermal Conduction

Heat conduction is the transfer of thermal energy from a region at higher temperature (T_h) to a part at lower temperature (T_l) of a body by microscopic collisions of atoms, molecules and electrons, and it occurs in all phases - solid, liquid, and gas. For a uniform object (slab, cylinder, etc.) with parallel opposite faces having surface area A separated by length L and maintained at constant temperatures T_h and T_l , respectively, with the help of relevant heat reservoirs, the time rate of thermal energy transfer from the hotter face to the colder face, $\frac{dQ(t)}{\mathrm{dt}}$, in the steady state, is given by the Fourier law:

$$\frac{dQ(t)}{\mathrm{dt}} = kA\frac{T_h - T_l}{L}, \tag{80}$$

where k is thermal conductivity of the constituent material. L/k is usually referred to as thermal resistance of the body, and the quantity $\frac{1}{A}\frac{dQ(t)}{\mathrm{dt}} = \frac{d\Phi(t)}{\mathrm{dt}}$ is called thermal flux density. In the framework of FC, Eq. (80) reads

$$D^\alpha Q(t) = k_F A \frac{T_h - T_l}{L}, \quad 0 < \alpha \leq 1. \quad (81)$$

Here, k_F is fractional thermal conductivity. Note that for a specific system, the right side of Eq. (81) is constant. Comparing this equation with Eq. (35), and using the relevant correspondences, we have from Eq. (38), for the quantity of thermal energy transferred in time t,

$$Q(t) = k_F A \frac{T_h - T_l}{L} \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad (82)$$

where we have taken $Q(0)$= 0. Obviously, the formula for $Q(t)$ given in books is recovered from Eq. (82) by putting α = 1. By choosing appropriate values of various parameters for a block of, say, plywood or PVC, we can have magnitude of $k_F A \frac{T_h - T_l}{L}$ equal to unity so that the plots for $Q(t)$ as function of time for different α values will be identical to those in Fig. 1, with ordinate at t = 0 as 0 rather than 1.

5.6 Newton's Law of Cooling

According to Newton's law of cooling, the rate of heat loss of a body at any time t, ($dQ(t)/dt$), is directly proportional to the difference in its instantaneous temperature, $T(t)$, and that of its surrounding environment (T_e) provided the temperature difference $[T(t) - T_e]$ is small and the nature of heat transfer mechanism is unchanged. Thus, $\frac{dQ(t)}{\mathrm{dt}} = -\xi[T(t) - T_e]$, where ξ is temperature-independent heat transfer factor determined by the area and nature of the body surface, and negative sign takes care of the fact that heat is being lost. The surrounding is assumed to be such that its temperature T_e is not changed by the heat gained from the body. Since $Q(t)$ equals product of heat capacity and temperature $T(t)$ of the body, we can write the above equation as

$$\frac{dT(t)}{\mathrm{dt}} = -\mu[T(t) - T_e], \quad (83)$$

with μ as cooling constant. The corresponding fractional differential equation for rate of fall in body temperature reads

$$D^\alpha T(t) = -\mu_F[T(t) - T_e], 0 < \alpha \leq 1. \quad (84)$$

Taking LT on both sides, rearranging the terms to find an expression for LT of $T(t)$ and then proceeding as in subsection 5.2, we finally get

$$T(t) = T_e + [T(0) - T_e] E_{\alpha,1}(-\mu_F t^\alpha). \quad (85)$$

For special case $\alpha = 1$, this becomes

$$T(t) = T_e + [T(0) - T_e]\, e^{-\mu t}, \quad (86)$$

as is obtained by solving Eq. (83).

It may be mentioned that Mondol et al [12] have investigated the Newton's law of cooling in the framework of FC by performing meticulous experiments on different samples of water, mustard oil, and mercury. They analysed their data to determine the best fit plots in respect of the FD parameter, using an appropriately determined value of cooling parameter μ_F. In order to discuss our expression (85), we have used their results for 80 ml water sample with initial temperature $T(0) = 100^0$ C and surrounding temperature T_e= 23.5^0 C. For this case, they obtained the best fit for μ_F= 0.109 and $\alpha = 0.79$. Employing their values for various parameters, we have drawn plots for $T(t)$ as function of t for $\alpha = 0.5$, 0.75, 0.79, 0.85. and 1.0; Fig. 8. Also shown in this figure are some experimental points extracted from the relevant plots in ref. [12]. A perusal of these plots clearly brings out the inadequacy of integer order Newton's law (α = 1.0), and highlights the correctness of the findings for α = 0.79. In fact, Mondol et al [12] found that their experimental results for all the liquid samples studied by them could be interpreted in terms of Eq. (85) using $\alpha < 1$, indicating the importance of the memory effects in cooling of liquids.

5.7 Ohm's Law and RC, RL Circuits

Ohm's Law, which is one of the most basic laws of electrical theory, states that the instantaneous electric current $i(t)$ (i.e., the time rate of change of electric charge) flowing through a conductor is directly proportional to the potential difference, V(t), across it at that instant of time. The constant of proportionality, reciprocal of the resistance (R) of the conductor depends on nature of its material, length, area of cross section and temperature. Thus, it reads $i(t) = \frac{\mathrm{dq}(t)}{\mathrm{dt}} = \frac{V(t)}{R}$. The fractional version of this law becomes $R_F D^\alpha q(t)$=$V(t)$, $0 < \alpha \leq 1$.

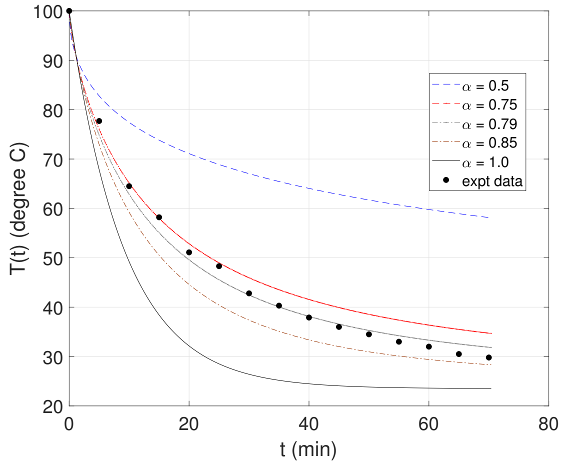

Figure 8: Time dependence of temperature for different values of fractional order derivative parameter α for water sample with $T(0) = 100^0$ C, T_e= 23.5^0 C, and μ_F= 0.109 [12]. The sample experimental points shown here have been obtained from figure 7 in the research paper of Mondol et al [12].

Here, R_F is fractional resistance having units Ohm $s^{\alpha-1}$.

Ohm's Law, which is one of the most basic laws of electrical theory, states that the instantaneous electric current $i(t)$ (i.e., the time rate of change of electric charge) flowing through a conductor is directly proportional to the potential difference, V(t), across it at that instant of time. The constant of proportionality, reciprocal of the resistance (R) of the conductor depends on nature of its material, length, area of cross section and temperature. Thus, it reads $i(t) = \frac{\mathrm{dq}(t)}{\mathrm{dt}} = \frac{V(t)}{R}$. The fractional version of this law becomes $R_F D^\alpha q(t) =$

$V(t)$, $0 < \alpha \leq 1$. Here, R_F is fractional resistance having units Ohm $s^{\alpha-1}$.

Now, if we consider a series circuit of a resistor having resistance R and a capacitor of capacitance C (so that instantaneous potential drop across this is $q(t)/C$) connected to a constant source of potential difference V (a battery), then

$$R\frac{\mathrm{dq}(t)}{\mathrm{dt}} + \frac{q(t)}{C} = V, \quad (87)$$

which in the framework of FC reads

$$R_F D^\alpha q(t) + \frac{q(t)}{C} = V, 0 < \alpha \le 1. \quad (88)$$

Proceeding as in subsection 5.2 to solve this fractional differential equation, we get for electric charge on the capacitor,

$$q(t) = \mathrm{VC} + [q(0) - VC]E_{\alpha,1}(-\frac{t^\alpha}{\tau_F}), \quad (89)$$

where $\tau_F = R_F C$ is the fractional capacitive time constant of the circuit. If the initial charge on the capacitor is zero, then

$$q(t) = \mathrm{VC}\left[1 - E_{\alpha,1}\left(-\frac{t^\alpha}{\tau_F}\right)\right]. \quad (90)$$

In contrast, if the capacitor is initially charged and is allowed to discharge through a resistor, so that $V = 0$, then Eq. (89) yields

$$q(t) = q(0)E_{\alpha,1}(-\frac{t^\alpha}{\tau_F}). \quad (91)$$

For $\alpha = 1$, the conventional equations for charging and discharging of the capacitor through the resistance are recovered from the preceding equations, with $\tau = RC$ as the capacitive time constant.

A comparison of Eqs. (50) and (89) shows that the latter can be obtained from the former by replacing $v(t)$ by $q(t)$, $\frac{\mathrm{mg}}{\gamma}$ by VC, and $\frac{\gamma}{m_F}$ by $\frac{1}{\tau_F}$; and Eq. (90) is special case of Eq. (50) with $v(0) = 0$. Consequently, the plots for charging of a capacitor obtained from Eq. (90) with relevant values of different parameters, will be analogous to those in Fig. 3.

We can similarly discuss the RL series circuit (connected to a battery of voltage V), wherein capacitor is replaced by an inductor with inductance L, for which the magnitude of instantaneous potential drop is given by $L\frac{\mathrm{di}(t)}{\mathrm{dt}}$, and the fractional differential equation takes the form $L_F D^\alpha i(t) + Ri(t) = V$. Here, L_F is the fractional inductance, and we talk about growth and decay of current. Furthermore, the characteristic time involved, the fractional inductive time constant, is given by $\tau_F = \frac{L_F}{R}$.

5.8 Random Decay / Growth Processes and Relaxation Phenomena

Many times, we come across situations where the constituents of a sample undergo random decay or growth in the sense that we cannot predict which entity will change the next regardless of how long it has existed. As typical examples, we may mention the decay or disintegration of radioactive nuclei and growth of bacteria. In such a case, if the number of identical constituents at an instant of time t is $N(t)$, and the happening of an event is independent of the preceding one (no memory effect), then the time rate of change is given by

$$\frac{\mathrm{dN}(t)}{\mathrm{dt}} = \mp\lambda N(t). \quad (92)$$

Here, negative sign indicates decay while the positive sign pertains to growth. λ (> 0) is called decay or growth constant and gives the probability of occurrence of that process. This interpretation demands that $N(t)$ must be a sufficiently large number. The corresponding fractional differential equation for the decay or growth process reads

$$D^{\alpha}N(t) = \mp \lambda_F \, \mathrm{N}(t), \; 0 < \alpha \le 1, \quad (93)$$

with λ_F as fractional decay or growth parameter. Obviously, this formula incorporates memory effect in the decay or growth process. Once again, following the usual procedure, we get

$$N(t) = N(0)E_{\alpha,1}(\mp \lambda_F t^{\alpha}). \quad (94)$$

$T_F = 0.693/\lambda_F$ defines half-life or doubling time for the entity undergoing decay or growth, with memory effects. Furthermore, using $P(t)$ to denote electric polarization in a complex disordered system (where memory effects of excitations are important) and introducing $\tau_F = 1/\lambda_F$, we have from Eq. (94),

$$P(t) = P(0)E_{\alpha,1}\left(-\frac{t^{\alpha}}{\tau_F}\right), \quad (95)$$

which is relaxation equation for the system characterized by fractional relaxation time τ_F. In fact, this equation can be generalized to describe the relaxation or return of any perturbed system to its equilibrium state as, for example, in nuclear magnetic resonance, vibrational energy relaxation, structural relaxation, etc.

Note that for $\alpha = 1$, Eqs. (94) and (95), respectively, reduce to

$$(N(t) = N(0)e^{\mp \lambda t},$$

and

$$P(t) = P(0)e^{-t/\tau}. \quad (96)$$

The first equation with negative sign is the well-known Rutherford-Soddy formula for radioactive decay of nuclei with disintegration constant λ, and the second expression is the Debye relaxation relation with relaxation time τ, used for simple dielectric materials.

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