# Solitons: Waves with many Attributes of Particles*

Vishwamittar[1]

[1] Retired from Department of Physics, Panjab University, Chandigarh 160014, India.

Contact Address: #121, Sector-16, Panchkula-134113 (Haryana).
vm121@hotmail.com

**The article is dedicated with warm regards to Prof. C. G. Mahajan, Prof. K.S. Harchand, Prof. K.C. Mittal, Prof. Vijendra K. Agarwal, and (late) Prof. R.S. Sud, and their families.*

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## Abstract

The purpose of this article is to expose the students and other readers to the wonderful world of solitons – the self-sustaining solitary waves. Beginning with a brief review of the history of their discovery as waves on water surface and their modeling by the Korteweg-de Vries nonlinear partial differential equation, and characterization as particles, we have given a concrete definition for these. A simple soliton solution for the afore-mentioned equation has been elucidated. Also included is a summary of Sine-Gordon and Nonlinear Schrӧdinger equations. The question 'why care for solitons?' has been answered by giving an overview of multifaceted theoretical and practical applications of its concepts in various branches of science, particularly physics. Effort has been made to keep the presentation as elementary as possible omitting some mathematical subtleties of the subject.

## 1 Introduction

When we read or hear the word 'wave', the immediate thing that comes to our mind is 'the wave moving on the surface of water'. A stone thrown into still water of a pond creates a disturbance that travels radially outwards in all directions from the point of hitting while the water particles on the surface vibrate up-and-down. Thus, as the wave propagates away from the point of its origin, the water particles remain where they were (a cork or a paper boat placed on the surface shows only up-and-down oscillations but no forward motion) and only energy is transported outwards.

A wave is a continuous disturbance from the state of equilibrium that travels from one region of space to another and transports energy / information without any translational movement of the intervening medium. The properties which characterize a wave and distinguish one from another are velocity, amplitude, angular frequency, and wavelength. Some well-known examples of waves are transverse waves on vibrating strings (which form the basis of musical instruments like veena, sarangi, violin, etc.), longitudinal or pressure waves in a gas, voltage and current waves along an electrical transmission line, electromagnetic waves (with light and radio waves as typical examples) in the free space / vacuum or a material medium, etc. Interestingly, irrespective of their diverse individual properties, these waves are treated by common mathematical formalism [1].

The partial differential equation (PDE) describing a progressive wave, travelling along $x$ – direction with disturbance function $u(x,t)$ at point $x$ at instant of time $t$, reads

$$\frac{\partial^2 u(x,t)}{\partial t^2} = v^2 \frac{\partial^2 u(x,t)}{\partial x^2}. \qquad (1)$$

This celebrated wave equation was first introduced and solved in a general way by d'Alembert in 1747, while developing a mathematical model of a vibrating string. Here, $v$ is the velocity of propagation of the wave and is also called phase velocity of the wave. $u(x,t)$ represents transverse displacement in a string or water wave, pressure in a sound wave in air, voltage or current in an electrical transmission line (where Eq. (1) is called telegraphist's equation) and so on. Note that Eq. (1) is a linear partial differential equation so that superposition principle holds good. Accordingly, if $u_1(x,t)$ and $u_2(x,t)$ are solutions of this second-order differential equation, then any linear combination of these functions is also a solution. From physics point of view this means that if two (or more) different waves are present in a medium, the disturbance at any point at any given time is the sum of the disturbances separately produced by these individual waves.

It is common practice to denote partial derivatives of a function $u\,(x,t)$ with respect to time and position coordinate by using $t$ and $x$ as subscripts of $\partial$ and $u$, and to suppress the explicit dependence on these variables. Thus, $\frac{\partial u(x,t)}{\partial t} \equiv \partial_t u \equiv u_t$ , $\frac{\partial u(x,t)}{\partial x} \equiv \partial_x u \equiv u_x$, and so on. In these short-hand notations, the classical wave equation, Eq. (1), can be written as

$$\partial_t^2 u - v^2 \partial_x^2 u = u_{\text{tt}} - v^2 u_{\text{xx}} = 0. \qquad (2)$$

We shall use the second abbreviation in this article. It may be added that waves are quite commonly observed in higher spatial dimensions and the preceding equation is modified to read

$$u_{\text{tt}} - v^2 \nabla^2 u = 0, \qquad (3)$$

where $\nabla^2$ is the Laplace operator in the chosen coordinate-system.

The wave travelling in $+x$ – direction, sometimes called the forward wave, is represented by the implicit function $u\,(x,t) =$

$f_1(x - vt)$ while the one moving in $-x$ – direction (the backward wave) is given by $u(x,t) = f_2(x + vt)$. The arguments $x \mp vt$ are usually referred to as the characteristic variables. The profile of the wave is governed by the mathematical form of $f_1$ or $f_2$. Under ideal conditions, the waves described by $f_1$ and $f_2$ do not change their form as these propagate. Thus, the shape of the wave given by $f_1(x - vt)$ at time $t > 0$ will be the same as at $t = 0$ except that it is shifted to the right by an amount vt.

It may be added that if the function describing a progressive wave is such that the profile of the disturbance at time $t = 0$ is either a sine or a cosine function then it is known as a harmonic or a sinusoidal wave. Accordingly,

$$\begin{aligned} u(x,t) = & a\cos(k[x - vt]), \qquad (4) \\ & a\sin(k[x - vt]), \\ & \text{or} \\ & a\exp(ik[x - vt]) \end{aligned}$$

represent a harmonic wave of amplitude $a$, propagation constant or angular wave number $k$ (which gives periodicity in the space coordinate $x$), wavelength $\lambda = 2\pi/k$, and angular frequency $\omega = 2\pi v = \frac{2\pi v}{\lambda} = vk$. Therefore, Eq. (4) can also be written as

$$u(x,t) = a\cos(kx - \omega t) \qquad (5)$$

and so on. The argument $(kx - \omega t)$ is phase of the wave at point $x$ at time $t$. Obviously, for a specific point $x$ it changes linearly with time $t$. Of course, the phase of the wave can be generalized to read $(kx - \omega t + \theta)$, with $\theta$ as phase at $x = 0$, $t = 0$.

If the medium through which a wave is passing, is such that the phase velocity of the wave is the same for all frequencies (i.e., $v = \frac{\omega}{k}$ = constant, independent of both $\omega$ and $k$), then it is called a non-dispersive or dispersion less medium. On the other hand, a medium is said to be dispersive if the wave velocity is different for different frequencies ($\frac{\omega}{k} \neq$ constant). Note that free space is non-dispersive medium for light waves while glass is a dispersive medium for these.

Now, if we consider an arbitrary pulse (a one-time disturbance or a wave of very short duration), which is a linear superposition of many harmonic waves with different angular frequencies, it will travel without deformation in its profile in a non- dispersive medium as all the constituent waves move with the same speed. However, in a dispersive medium, the phase velocities of the component harmonic waves are different so that the fast-moving constituents go ahead, and the slow ones lag behind. Consequently, the pulse changes its shape as time evolves and will spread out or disperse as it moves (leading to decrease in its amplitude and increase in width). In this case, we talk about group velocity $v_g = d\omega/dk$. It is this velocity with which energy is transported by the pulse or any wave comprising different frequencies. The expression, such as $\omega = k - k^3$ or $\omega = \omega_0|\sin(\frac{k}{2})|$, giving the variation of $\omega$ as a function of $k$ is known as a dispersion relation.

Before proceeding further, it is worthwhile to point out that if we redefine time such that $t^{'} = vt$, then $u_t = \frac{\partial u(x,t^{'})}{\partial t^{'}} \frac{\partial t^{'}}{\partial t} = vu_{t^{'}}$ and, similarly, $u_{\rm tt} = v^2 u_{t't'}$. Accordingly, Eq. (2) is transformed to read $u_{t't'} - u_{\rm xx}= 0$. Replacing $t'$ by $t$ and keeping in mind that the rescaled time has dimension of length rather than time, we can rewrite the wave equation, as $u_{\rm tt} - u_{\rm xx} = 0$. A comparison of this equation with the original equation, viz. Eq. (2), shows that the envisaged transformation is equivalent to taking $v = 1$. Of course, dimension of $u$ in both the equations is the same.

It is pertinent to note that like the linear differential equation describing simple harmonic oscillator, the wave equation, Eq. (1) or (2), is obtained by assuming the amplitude of wave to be small. As such, it is an idealized model for the one- dimensional wave motion. However, if the derivation is made for more realistic situations, which necessarily involve nonlinearity, we get wave equations involving dispersive as well as nonlinear terms. One such nonlinear partial differential equation (NLPDE) was derived by Korteweg and de Vries (usually abbreviated as KdV) in 1895 to describe the propagation of waves in one-dimension on the surface of a shallow canal assuming the flow to be inviscid, incompressible, steady and irrotational. In its standard dimensionless form, as commonly used in the current literature, it reads [2, 3, 5, 8]

$$u_t - 6uu_x + u_{\rm xxx} = 0. \qquad (6)$$

Here, $t$ and $x$ are normalized time and normalized coordinate in the direction of wave propagation, respectively. If in any problem similar equation of evolution turns out to be of different form, it can be transformed into this standard form of the KdV equation by using an appropriate scale. Here, the first term gives time evolution of the disturbance proceeding in $+x$ – direction. The second term in this equation is nonlinear, which leads to steepening or narrowing of the wave. Also, because of the presence of nonlinear term, the principle of superposition of solutions does not hold good. This, in turn, makes wave structure robust in interactions / collisions with other wave structures. The third term is the dispersion part and will give rise to a nonlinear relationship between $\omega$ and $k$. In fact, the KdV equation is the simplest NLPDE that incorporates both nonlinearity and dispersion.

Eq. (6) admits a solution of the form $u\,(x,t) = A\,\text{sech}^2\,(x,t,A)$ indicating presence of amplitude in the argument. This represents a bell-shaped profile, which describes a solitary wave first observed by Russell in 1834. This feature of the KdV equation and its modified form has been found to be very useful in the study of waves in elastic rods, liquid-gas bubble mixtures, plasmas, anharmonic lattices, etc., besides the water waves.

In this article, we delineate upon the fascinating and interesting topic of solitary waves from pedagogic point of view at reasonably basic level [2-10]. In Section 2, we give an overview of their discovery, development of the subject and nomenclature as solitons. This is followed by Section 3 where a precise definition of solitons is given. Section 4 is devoted to description of a simple soliton-solution of the KdV equation which is being used as a prototypical example of exactly solvable soliton-bearing model. This approach involves easily understandable mathematics and, still, makes the concept quite transparent. Also included are some remarks regarding various solutions of this equation. Then we move on to Section 5 to give a brief information about two other NLPDEs leading to different flavours of solitons. Section 6 summarizes versatile applications of the solitons to a wide variety of systems in diverse fields. We close the article in Section 7 by making some general comments.

## 2 A Historical Account of the Discovery of Solitary Waves and Growth of the Subject

We shall not be able to do justice to all the spectacular developments in vast subject of solitons with an elegant history of nearly two centuries and shall concentrate mainly on those contributions that had a larger influence on the overall progress, particularly from physics point of view.

It was in 1830's that a Scottish civil engineer and naval architect named John Scott Russell, with a view to develop an efficient design for canal boats, performed experiments on moving boats in Edinburgh-Glasgow canal to find relation between their shape, speed, and the force needed to push them. One day in August 1834, this young man (then 26 years old) was observing the motion of a boat that was being rapidly drawn along a narrow channel by a pair of horses. He found that when the boat suddenly stopped, the moving water collected around it in a state of violent agitation and then abruptly it rolled forward with great velocity of about 13 - 14 km / hr in the form of a nearly 9 m long and 30 - 50 cm high smooth and well-marked accumulation of water. This heap travelled on the surface of water without any change in its profile or speed till it was lost in the windings of the channel after a follow up of about 2 km. He called this singular wave 'the wave of translation'. Obviously, this discovery was essentially a random happenstance.

Impressed by this unexpected observation, Russell carried out extensive meticulous experiments about the nature of these waves of elevation in many canals, rivers, lakes, and in a large wave tank in his back garden. He concluded that this wave motion was unique and quite different from other types of oscillatory motions – the speed depends on its amplitude and the depth of water, and they never merge. Therefore, he started referring to them as 'solitary waves' in the sense that this wave had only a single protuberance traveling without any change in its shape, size, or speed. Treating it as a gravity wave, he found that speed of the wave on a water surface with undisturbed depth $h$ is given by $v = \sqrt{g(h+a)}$ , where $a$ is amplitude of the wave. This showed that the larger the amplitude of a wave higher its speed – a miraculous nonlinear effect.

However, he could not convince his contemporaries, particularly mathematicians, about the importance and even novelty of these waves mainly because his findings were at variance with the then accepted theories of hydrodynamics and he himself could not give an analytical formalism. What was strikingly surprising and unusual about this wave and was not appreciated by the scientists at that time is: A coherent hump of water was formed out of turbulence produced by sudden stopping of boat in shallow water and this protrusion maintained its characteristics over quite a long distance in contrast with the normal behavior of water waves that spread out and disappear after travelling over reasonably short distances.

Despite this situation, Russell's work was followed by Stokes' efforts in 1847 to get some theoretical interpretation and by Boussinesq's (1871) and Rayleigh's (1876) successful explanation of the nature of these waves. They used Euler's equations of motion for an inviscid, incompressible fluid and not only obtained Russell's formula for speed but also an expression for the wave profile, reading

$$u\,(x,t) = a\,\mathrm{sech}^2\,[A\,(x-vt)]. \tag{7}$$

Here, A is a parameter that depends on the amplitude $a$ and the height h of water surface from the base of the canal. This expression is strictly true for $a \ll h$. However, these scientists did not derive or write the differential equation satisfied by the above expression for $u\,(x,t)$. This task was done by Dutch mathematician Korteweg and his student de Vries in 1895 who obtained the remarkable Eq. (6) and derived various wave properties which were similar to those observed by Russell in different experiments, though they did not refer to the work done by him. Not only this, it also seems that even they themselves did not realize the importance of their finding as they did not pursue it further. Continuing the narration of the history, it may be mentioned that in 1955, Fermi, Pasta, Ulam, and Tsingou, working at one of the world's earliest computers (the MANIAC machine), performed numerical investigation of heat transfer in a solid modeled by a one-dimensional lattice consisting of equal mass anharmonic oscillators. They observed that there was a periodic recurrence in the distribution of energy rather than expected equipartition of energy among the modes. This astounding result and the fact that the system considered by these scientists was closely related to discretization of the KdV equation, prompted Zabusky and Kruskal (1965) to undertake the initial value problem for the KdV equation [11]. Pursuing insightful numerical simulations, they (i) mimicked the Russell's solitary waves; (ii) explained the odd results of Fermi, Pasta, Ulam, and Tsingou; and (iii) found that when two or more of the KdV solitary waves interact or collide with each other they neither break up nor disperse and rather emerge out preserving their individual shapes and velocities as if there was no interaction. These do undergo a small change in their phase on collision.

Keeping in view their last- mentioned novel finding that assigned remarkable corpuscular or particle-like characteristic to these waves, they coined the term 'soliton' for these solitary waves to emphasize its kinship with electron, photon, phonon, etc. that behave like both particle and wave. In fact, it was this milestone work that brought the KdV equation into limelight after being in obscurity for nearly seven decades. However, the first rigorous analytical solution to this famous equation reading $u(x,t) = -B\,\mathrm{sech}^2\left[\sqrt{\frac{B}{2}}\,(x-2Bt-x_0)\right]$ was given by Gardner and coworkers in 1967 [5,12]. The method developed by them involves formulating a scattering problem with desired solution as potential and solving this as a first step. The outcome of this solution is then used to reconstruct $u(x,t)$. This technique is referred to as Inverse Scattering Method. They also obtained the general multi-solitons or $n$-solitons solution for the KdV equation. The salient feature of this method lies in the fact that it provides exact solution for nonlinear wave equations by linear techniques and is useful in discovering solitons. Later, this ingenious approach together with its generalizations and the novel method put forward by Hirota (1971) for obtaining multi-soliton solutions, provided powerful tools for solving many physically interesting NLPDEs and, thus, for studying solitons. However, we shall not dwell on details of these techniques or other methods developed for solving the soliton-bearing equations as these are too technical in nature. In the meantime, Toda (1967) reported existence of a soliton in a discrete, integrable system, which is now called Toda lattice.

These developments opened up fascinating vistas, and established study of solitons or solitary waves as a vibrant and flourishing topic of research among mathematicians, physicists, engineers, and others. On one hand, this boom led to discovery of numerous soliton-bearing nonlinear evolutionary PDEs in one or more space-dimensions and thus adding to mathematical richness of theory of solitons. On the other hand, solitons became objects of immense physical importance. In fact, solitons play same role in the description of nonlinear systems as harmonic waves in the linear systems. Consequently, lot of effort has been directed at exploiting fecundity of applications of this concept in different branches of science. However, before going ahead, we first define solitons in Section 3.

## 3 Defining a Soliton

Strictly speaking solitons are such solutions of the NLPDEs that (i) do not change profile while travelling nor do they disperse, implying complete stability; (ii) survive collisions, emerging unblemished; (iii) cannot be constructed as a superposition of harmonic waves; and (iv) the speed of the wave profile depends on its amplitude. Thus, solitons are self-reinforcing, non-dissipative, and persistent solitary waves of finite amplitude, and are indubitably nonlinear entities. These propagate undistorted over long distances and maintain their speed and shape upon collision / interaction with other such waves.

However, the term 'soliton' has been used by scientists in a relatively loose manner for the objects which do not necessarily fulfil all the above requirements. It is, in a way, used to signify a spatially compact, finite field energy configuration which may or may not be time dependent. This term has also been adopted to cover a large class of solitary excitations that are localized in space-time and though long-lived, are only metastable. Thus, the condition of these being perfectly stable is relaxed. In this sense, some of the soliton solutions can be identified as elementary excitations. Such moderation of the definition has made the realm of usage of the theory of solitons quite vast.

## 4 Rudimentary Solution of the KdV Equation

Guided by the approach presented by Drazin and Johnson [2], to obtain a travelling or progressive wave solution to the KdV equation, Eq. (6), we introduce a new variable or parameter $\eta = x - vt$, which represents the position in a reference frame moving with the wave with speed $v$. Note that $\frac{\partial \eta}{\partial x}$ = 1 and $\frac{\partial \eta}{\partial t} = -v$. Also, the solution can be written as $f(\eta) \equiv f$ in place of $u(x,t)$ and it represents a wave travelling with speed $v$ in the original coordinate system. Now,

$$u_t = \frac{\partial u}{\partial t} = \frac{df}{d\eta}\frac{\partial \eta}{\partial t} = -vf',$$

$$u_x = \frac{\partial u}{\partial x} = \frac{df}{d\eta}\frac{\partial \eta}{\partial x} = f',$$

and

$$u_{\text{xxx}} = \frac{\partial^3 u}{\partial x^3} = f'''.$$

Making these substitutions into Eq. (6), we get

$$-vf' - 6ff' + f''' = 0. \quad (8)$$

Obviously, the NLPDE has been transformed into an ordinary differential equation with the nonlinear and dispersive terms intact. Integrating the above differential equation with respect to single variable $\eta$, we have

$$-vf - 3f^2 + f'' = C_1, \quad (9)$$

where $C_1$ is arbitrary constant of integration. Multiplying with $f'$ on both sides of this equation and integrating again, we obtain

$$-v\frac{f^2}{2} - 3\frac{f^3}{3} + \frac{(f')^2}{2} = C_1 f + C_2. \quad (10)$$

Here, $C_2$ is second arbitrary constant. Eq. (10) can be rewritten as

$$(f')^2 = 2\{f^3 + \frac{v}{2}f^2 + C_1 f + C_2\} \equiv 2\,F(f). \quad (11)$$

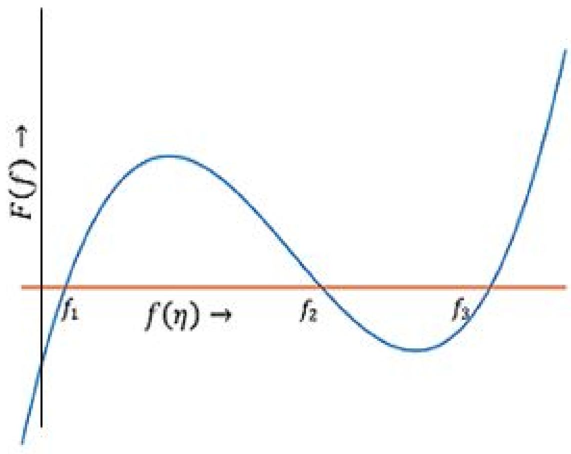


Figure 1: An arbitrary plot showing typical dependence of $F(f)$ on $f$ as per Eq. (11).

This gives us $f^{'}$ in terms of a cubic polynomial in $f$, where $C_1$ and $C_2$ are determined by the initial conditions satisfied by the KdV equation.

Now, $f(\eta)$ being solution of wave equation, it represents a (classical) wave displacement, and, therefore, it must be real (so that it can be observed) and finite or bounded. This, in turn, implies that $f^{'}$ too is real so that $(f^{'})^2 \geq 0$ and hence $F(f) \geq 0$. Thus, only those $f(\eta)$ are physically acceptable for which $F(f)$ is non-negative. Since $F(f)$ is a cubic polynomial, it will have three real-valued zeros defined by $f^3 + \frac{v}{2}f^2 + C_1 f + C_2 = 0$. Let these be $f_1$, $f_2$, and $f_3$ and, in general, such that $f_1 < f_2 < f_3$. Of course, sometimes two or all the three zeros may coincide with each other. Note that for a cubic polynomial with the coefficient of the cubic term as unity, the sum of its zeros equals negative of the coefficient of the square term. Thus,

$$f_1 + f_2 + f_3 = -\frac{v}{2}. \qquad (12)$$

Since $v$ is speed of propagation of the wave, it will be positive along the $+x$ – direction and this demands that

$$f_1 + f_2 + f_3 < 0. \qquad (13)$$

Obviously, $f_1$ will certainly be negative and the signs of $f_2$ and $f_3$ may be negative or positive depending on the values of $v$, $C_1$ and $C_2$.

For extremely large $|f|$, $F(f)$ is governed by $f^3$, and, therefore, $F(f)$ is negative for negative large values of $f$ and is positive for positive large magnitudes of $f$. Since $f_1$ and $f_3$ are, respectively, the lowest and the largest zeros of $F(f)$, it will be negative for $f < f_1$ and it will be positive for $f > f_3$. So, at the zero $f_1$, $F(f)$ goes from negative values to positive values, and at the zero $f_3$, it must again go from negative to positive values. Accordingly, at the zero $f_2$, sign of $F(f)$ values changes from positive to negative; Fig. 1. Thus, $F(f)$ and hence $(f^{'})^2$ is positive for $f_1 < f < f_2$ and for $f > f_3$. But it is bounded only for $f_1 < f < f_2$, Therefore, acceptable solution $f(\eta)$ must lie between $f_1$ and $f_2$, which must be distinct.

Since $f_1$, $f_2$, and $f_3$ are zeros of $F(f)$, we can express it as product of three factors:

$$F(f) = (f - f_1)(f - f_2)(f - f_3). \qquad (14)$$

This together with Eq. (11) gives us

$$\frac{df}{d\eta} = \pm[2(f - f_1)(f - f_2)(f - f_3)]^{1/2}. \qquad (15)$$

Confining ourselves to the region $f_1 < f < f_2$ (and the corresponding $\eta$ values $\eta_1 < \eta < \eta_2$), we get from Eq. (15)

$$\int_{\eta_1}^{\eta} d\eta = \pm \int_{f_1}^{f} \frac{\mathrm{dg}}{[2(g-f_1)(g-f_2)(g-f_3)]^{1/2}}. \tag{16}$$

We have used $g$ as variable in the integral on the right-hand side as $f$ is being taken as upper limit. Now, we substitute $g = f_1 + (f_2 - f_1)\sin^2\theta$ so that lower limit $g = f_1$ corresponds to $\theta = 0$, and the upper limit $g = f$ implies

$$f = f_1 + (f_2 - f_1)\sin^2\Theta, \tag{17}$$

where $\Theta$ is upper limit value of $\theta$. Making these substitutions together with $dg = 2(f_2 - f_1)\sin\theta\cos\theta\, d\theta$ on the right-hand side of Eq. (16), simplifying the resulting expression, and using the fact that left-hand side equals $\eta - \eta_1$, we finally obtain

$$\begin{aligned}\eta &= \eta_1 \pm \sqrt{\frac{2}{f_3 - f_1}} \int_{\theta}^{\Theta} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}} \\ &= \eta_1 \pm \sqrt{\frac{2}{f_3 - f_1}}\mathtt{w} \quad (say).\end{aligned} \tag{18}$$

Here,

$$m = \frac{f_2 - f_1}{f_3 - f_1}, \tag{19}$$

such that $0 \leq m \leq 1$. Also,

$$\mathtt{w} = \int_{\theta}^{\Theta} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}, \tag{20}$$

is incomplete elliptic integral of first kind with parameter $m$.

Now, we define a new pair of functions corresponding to $\mathtt{w}$:

$$\mathrm{sn}\,\mathtt{w} \equiv \mathrm{sn}\,(\mathtt{w}|m) = \sin\Theta, \tag{21a}$$

$$\mathrm{cn}\,\mathtt{w} \equiv \mathrm{cn}\,(\mathtt{w}|m) = \cos\Theta, \tag{21b}$$

These are, respectively, called the Jacobi elliptic sine (snoidal) and Jacobi elliptic cosine (cnoidal) functions. $\Theta$ is usually referred to as Jacobi amplitude.

Note that for $m = 0$, $\mathtt{w} = \int_0^{\Theta} d\theta = \Theta$, so that

$$\mathrm{sn}\,\mathtt{w} \equiv \mathrm{sn}\,(\mathtt{w}|0) = \sin\Theta = \sin\mathtt{w}, \tag{22a}$$

$$\mathrm{cn}\,\mathtt{w} \equiv \mathrm{cn}\,(\mathtt{w}|0) = \cos\Theta = \cos\mathtt{w}. \tag{22b}$$

Obviously, for $m = 0$, which happens when $f_2$ merges with $f_1$ from above, i.e., $f_2 \to f_1^+$, the functions sn $\mathtt{w}$ and cn $\mathtt{w}$ are periodic sin and cos functions, respectively.

On the other hand, for $m = 1$,

$$\begin{aligned}\mathtt{w} &= \int_0^{\Theta} \frac{d\theta}{\sqrt{1 - \sin^2\theta}} = \int_0^{\Theta} \sec\theta d\theta \\ &= \ln[\tan\Theta + \sec\Theta] \\ &= \ln\left[\frac{1 + \tan\left(\frac{\Theta}{2}\right)}{1 - \tan\left(\frac{\Theta}{2}\right)}\right]\end{aligned} \tag{23}$$

This, on simplification, yields

$$\tan\left(\frac{\Theta}{2}\right) = \tanh\left(\frac{\mathtt{w}}{2}\right) \tag{24}$$

which, in turn, gives $\sin\Theta = \tanh\mathtt{w}$ and $\cos\Theta = \mathrm{sech}\,\mathtt{w}$. These imply that

$$\mathrm{sn}\,\mathtt{w} \equiv \mathrm{sn}\,(\mathtt{w}|1) = \sin\Theta = \tanh\mathtt{w}, \tag{25a}$$

$$\mathrm{cn}\,\mathtt{w} \equiv \mathrm{cn}\,(\mathtt{w}|1) = \cos\Theta = \mathrm{sech}\mathtt{w}. \tag{25b}$$

Thus, for $m = 1$, the elliptic functions sn $\mathtt{w}$ and cn $\mathtt{w}$ are aperiodic tanh $\mathtt{w}$ and sech $\mathtt{w}$, respectively. Note that $m = 1$ if the zeros $f_2$ and $f_3$ of $F(f)$ coalesce to form a double zero ($f_2 \to f_3^-$ as $f_2 < f_3$) but are distinct from $f_1$.

After this digression, we come back to Eq. (17), replace $\sin^2\Theta$ by $1 - cos^2\Theta$, and get

$$f = f_2 - (f_2 - f_1)\cos^2\Theta. \qquad (26)$$

In view of Eq. (21b), this can be rewritten as

$$f = f_2 - (f_2 - f_1)\,\mathrm{cn}^2\left(\mathtt{w}|m\right), \qquad (27)$$

where

$$\mathtt{w} = \pm(\eta - \eta_1)/\sqrt{\frac{2}{f_3 - f_1}}, \qquad (28)$$

from Eq. (18). Since cn is an even function, we omit $\pm$ sign and express Eq. (27) as

$$f(\eta) = f_2 - (f_2 - f_1)\,\mathrm{cn}^2\left(\sqrt{\frac{f_3 - f_1}{2}}\{\eta - \eta_1\}\middle| m\right). \qquad (29)$$

This is called cnoidal wave solution of the KdV equation – generalization of the sinusoidal wave.

In the limit $m \to 0$, which is achieved when $f_2 \to f_1^+$, we use Eq. (22b) for cn $\mathtt{w}$ and then the double angle trigonometric identity $\cos^2\mathtt{w} = \frac{1}{2}(1 + \cos 2\mathtt{w})$, and get $f(\eta)$ in terms of cos function with $\frac{(f_2 - f_1)}{2}$ as coefficient. Thus, $f(\eta)$ describes an oscillatory cosine wave with amplitude $\frac{(f_2 - f_1)}{2}$, which, obviously, is quite small. It is found that the wave is dispersive in nature. This is low amplitude linear wave limit of the cnoidal solution. However, we shall not go into its further discussion.

Next, for the case $m = 1$, which is the most nonlinear limit, we use Eq. (25b) for cn $\mathtt{w}$ and put $f_2 = f_3$ into Eq. (29). Accordingly, we have

$$f(\eta) = f_3 - (f_3 - f_1)\,\mathrm{sech}^2(\{\sqrt{\frac{f_3 - f_1}{2}}\}\{\eta - \eta_1\}). \qquad (30)$$

Now, from the definition $\mathrm{sech}\,y = \frac{2}{e^y + e^{-y}}$, we note that $\mathrm{sech}\,y = 1$ for $y = 0$ and equals zero for $y \to \pm\infty$. Thus, $\mathrm{sech}^2(\{\sqrt{\frac{f_3-f_1}{2}}\}\{\eta - \eta_1\})$ =1 for $\eta = \eta_1$ and 0 for $\eta \to \pm\infty$. The corresponding values of $f(\eta)$ are $f_1$ and $f_3$, respectively. Since $f_1$ is necessarily negative and less than $f_3$, $f(\eta)$ has minimum value $f_1$ at $\eta = \eta_1$, and maximum value $f_3$ for $\eta \to \pm\infty$. In other words, if we plot a graph of $f(\eta)$ as function of $\eta$, this will be a wave profile with depression (upside-down) having value $f_1$ at $\eta = \eta_1$, and depth $f_3 - f_1$. However, as we are looking for a model to describe a waveform above the water surface, we consider $-f(\eta)$ rather than $f(\eta)$. Accordingly, $-f(\eta)$ represents a profile with $-f_1$ at $\eta = \eta_1$ as peak and $-f_3$ as minimum value for $\eta \to \pm\infty$. Consequently, we can identify $-f_1 - (-f_3) = f_3 - f_1$ as amplitude $a$ of the wave. Thus, Eq. (30) can be written as

$$-f(\eta) = -f_3 + a\,sech^2\{\sqrt{\frac{a}{2}}(\eta - \eta_1)\}. \qquad (31)$$

The velocity of this wave is given by

$$v = -2(f_1 + f_2 + f_3) = 2a - 6f_3, \qquad (32)$$

where we have used $f_2 = f_3$ and $f_3 - f_1 = a$ in Eq. (12). Furthermore,

$$\eta = x - vt = x + 6f_3 t - 2at. \qquad (33)$$

Note that $v$ is directly proportional to amplitude implying that the larger the amplitude the higher the speed. Also, for $v$ to be positive, the zero $f_3$ must be less than $\frac{a}{3}$.

Having obtained the solution, Eq. (31), for Eq. (6), and using Eq. (33), we can write

$$\begin{aligned} -u\,(x,t) &= -f_3 + a\,\mathrm{sech}^2\{\sqrt{\frac{a}{2}}\,(x - vt - \eta_1)\} \\ &= -f_3 + a\,\mathrm{sech}^2\{\sqrt{\frac{a}{2}}\,(x + 6f_3 t - 2at - \eta_1)\}. \end{aligned} \quad (34)$$

This describes a wave of elevation having nonperiodic bell-shaped profile of amplitude $a$ ($> 3\ f_3$), travelling with speed $v = 2a - 6f_3$, initial phase factor $-\sqrt{\frac{a}{2}}\ \eta_1$, and $-f_3$ as ambient or undisturbed or equilibrium level. The presence of $a$ in the argument of sech in Eq. (34) shows that the shape of the wave depends on amplitude in a complicated manner, which, in turn, implies that $-\ u\,(x,t)$ represents a nonlinear wave. From the first equality in Eq. (34) it is found that $-\ u\,(x,t) + f_3 = a$ if $x = vt + \eta_1$. Obviously, the peak appears at $x = \eta_1$ for $t = 0$ implying that $\eta_1$ can be taken to be 0 by using the location of the peak at $t = 0$ as reference for measuring $x$. Furthermore, $-\ u\,(x,t) + f_3 = a/2$ when $x_\pm = vt + \eta_1 + \sqrt{\frac{2}{a}}\ \ln\left(\sqrt{2} \pm 1\right)$. Taking the distance between the points at which the height of the wave above the ambient level is half the amplitude, as width of the profile, called full width at half maximum, we have $\Delta x \equiv x_+ - x_- = \sqrt{\frac{2}{a}}\ \ln\frac{\sqrt{2}+1}{\sqrt{2}-1}$. Thus, width of the profile is inversely proportional to $\sqrt{a}$. Combining this result with the statement after Eq. (33), we note that the wave of elevation described by Eq. (34) is such that taller the wave, narrower and faster it is. It is the solitary wave discovered by Russell and its plot is depicted in Fig. 2 for three values of $t$. Note that profile of the wave is the same for all the three $t$ values shown here and has $\Delta x = 3.94$.

As a follow up of the preceding discussion, suppose we launch two solitary waves having different amplitudes such that the one with smaller amplitude is leading. The wave with higher amplitude will have larger velocity so that as time passes it will come closer to the other wave, bump into it at some instant of time and ultimately overtake it. The end-result will be that the two waves pass through each other without losing their identity, i.e., they come out of the collision unscathed – a particle-like robustness. In fact, this aspect was also observed by Russell.

It may be mentioned that the actual solution, Eq. (30), representing a wave of depression rather than a wave of elevation, is a consequence of the negative sign of the nonlinear term in the standard form of the KdV equation, which has been solved here. Furthermore, the cnoidal wave solution, Eq. (29), is not the only possible solution to the KdV equation; other simple looking solutions have also been found. Besides, solutions leading to more than one soliton, have also been obtained.

It is worth emphasizing that the dispersion term $u_{xxx}$ in Eq. (6) gives rise to ten-

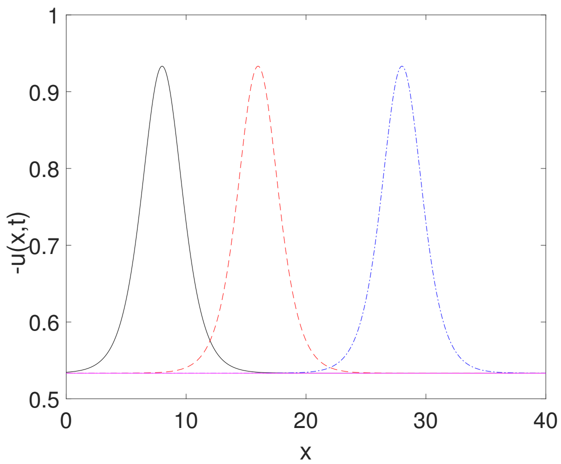


Figure 2: One-soliton solution of the KdV equation given by Eq. (34) at $t$ = 2.0 (black), 4.0 (red), and 7.0 (blue) for $a$ = 0.4; $v$ = 4.0, $\eta_1$= 0.

dency of flattening or spreading of the wave profile, while the nonlinear term $-6uu_x$ makes it steep and cohesive. The precise balancing of these two tendencies leads to 'no change in the shape' of the wave, i.e., the soliton solution. The KdV equation has been found to be very useful in modeling the dynamics of physical systems characterized by mild dispersion and weak nonlinearity.

By convention, the word 'soliton' is used for the wave profile with positive displacement (i.e., an elevation) and the envelope with negative displacement (i.e., a depression) is called an anti-soliton. Thus, $-\ u\,(x,t)$ given by Eq. (34) defines a soliton, while $u\,(x,t) \ =\ f_3 - a\,\mathrm{sech}^2(\sqrt{\frac{a}{2}}\,\{x+6f_3t-2at-\eta_1\})$ is an anti-soliton. When a soliton and corresponding anti-soliton collide with each other, the net displacement is zero and this is referred to as annihilation of soliton $-$ anti-soliton pair. However, generally these pairs collide and then separate.

## 5 Some Other Soliton-Bearing Nonlinear Partial Differential Equations

It has been pointed out in the preceding section that the solution of the KdV equation maintains its shape indefinitely because of exact cancellation of the spreading or broadening produced by the dispersive term and the narrowing effects of the nonlinear term. In fact, any NLPDE containing dispersive and nonlinear terms counterbalancing detrimental effects of each other will have soliton solution. Of course, these solitons can be distinctly different from the bell-shaped solitons of the KdV equation. Two such evolution equations having more than one soliton solution and finding wide range applications in physics, biology, and engineering, together with relevant brief comments, are listed below. While writing these NLDPEs, the variables involved are taken to be properly rescaled. In fact, these equations are more useful than the KdV equation, which has been discussed in detail not because it is the oldest but because it is the simplest in nature.

### 5.1 Sine-Gordon Equation

This NLPDE reads

$$u_{\mathrm{xx}} - u_{\mathrm{tt}} - \sin u = 0, \qquad (35)$$

with $\sin u$ as nonlinear term. The presence of $u$ as argument of sin implies that this equation describes angular disturbance expressed in radians. It was originally put forward by Bour in 1862 during the investigation of surfaces of constant negative curvature in 3-dimensional space. Later, it was rediscovered by Frenkel and Kontorova in 1939 in their seminal work on study of crystal dislocations, which are defects or irregularities in the crystal structure along some direction and can even be mobile. It was in 1962 that Perring and Skyrme obtained a 2-soliton solution for Eq. (35). Subsequently, 1- and 3- soliton solutions were also obtained. This equation drew lot of attention in 1970s onwards as it was found to be useful in explaining many physical phenomena and is the simplest NLPDE in a periodic medium. It is interesting to note that the name 'sine-Gordon equation' (SGE in short) has its origin in its resemblance to the well-known Klein–Gordon equation for a free particle in relativistic quantum mechanics, which reads $\sum_{j=x,y,z} \phi_{\rm jj} - \phi_{\rm tt} - \phi = 0$, in natural units $m = c = \hbar = 1$, and was discovered in 1926. Of course, the Klein-Gordon equation is a linear partial differential equation, which can be considered as special case of the SGE obtained by retaining only first term in the Taylor series expansion of $\sin u$.

One of the soliton solutions of Eq. (35) is

$$u\left(x,t\right) = 4\ \tan^{-1}\left[e^{\pm\frac{x-\alpha t - x_0}{\sqrt{1-\alpha^2}}}\right], |\alpha| < 1, \quad (36)$$

where $\alpha$ is normalized velocity of propagation of the solitary wave. The initial position $x_0$ can be easily taken as 0. Note that for finite constant value of $\alpha t$, $u\left(x,t\right)$ in Eq. (36) with positive exponent has values 0, $\pi$ and $2\pi$ rad for $x \to -\infty$, $x = \alpha t$ and $x \to \infty$, respectively. On the other hand, the corresponding values of $u\left(x,t\right)$ with negative exponents are $2\pi$, $\pi$ and 0 rad. Thus, the solution given by Eq. (36) is monotonically varying function of $x$, and is such that as $x$ increases from $-\infty$ to $\infty$ for fixed value of $t$, $u$ changes from 0 to $2\pi$ for positive exponent and from $2\pi$ to 0 for the negative exponent. The value of $u$ in both the cases is $\pi$ when $x = \alpha t$. This feature of the solution for the SGE is interpreted as following. Eq. (36) represents a twist or kink having same sign as that of the exponent. These two situations define soliton and anti-soliton, respectively, and are known as $2\pi$-kink and $-\ 2\pi$-kink (or antikink); Fig. 3. In the context of nonlinear optics, these are, respectively, referred to as $+2\pi$ pulse and $-2\pi$ pulse.

It is worth mentioning that a soliton solution is said to be topological if it has its origin in topological constraints and a twist with variation in the value of $x$ is an example of this situation. As such, the SGE kink is an iconic one-dimensional topological soliton while the Russell's water wave soliton is non-topological. In fact, the structure of a system is changed after the passage of a topological-soliton wave through this.

The soliton solutions of SGE and its modified versions find numerous and in-

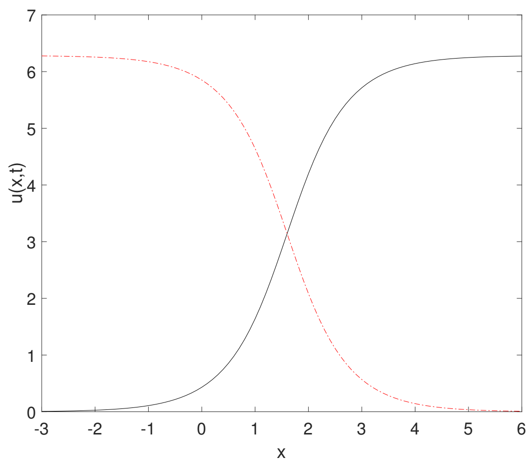


Figure 3: A sketch of the analytic solution $u\,(x,t)$ as function of $x$ for the Sine-Gordon equation for $\alpha$ = 0.8, $t$ = 2.0, and $x_0$=0. The black line represents a $2\pi$ – kink, while the dash-dot red line depicts antikink. These have value $\pi$ rad at $x = 1.6$.

valuable applications in condensed matter physics, nonlinear optics, biophysics, astrophysics, relativistic field theory, geophysics particularly seismic modeling, and in the description of mechanical transmission lines.

### 5.2 Nonlinear Schrödinger Equation

In 1968, Zakharov while studying nonlinear waves in the small amplitude approximation on the surface of a deep fluid introduced a NLPDE that can be written as

$$iu_t + u_{xx} \pm |u|^2 u = 0 \qquad (37)$$

This equation is referred to as Nonlinear Schrödinger equation (abbreviated as NLSE) because it looks like the highly acclaimed 1-dimensional time-dependent Schrödinger equation of non-relativistic quantum mechanics (i.e., $i\hbar\psi_t + \frac{\hbar^2}{2m}\psi_{xx} - V\psi = 0$), with nonlinear term $\pm|u|^2$ corresponding to potential $V$. Note that, the Schrödinger equation is a linear PDE and $\psi(x,t)$ is wavefunction of the particle assumed to be spinless. Of course, generally, the derivation of NLSE has nothing to do with quantum mechanics. The exact analytic solution of NLSE, obtained by Zakharov and Shabat in 1972 by using the inverse-scattering method, showed that these describe deep-fluid wave-envelope solitons which modulate a periodic sinusoidal wave. These findings were experimentally verified by Yuen and Lake in 1975. A different solitary wave solution to this equation was reported by Ma in 1979, and a rational-cum-oscillatory solution was presented by Peregrine in 1983. Note that $\pm\,|u|^2$ in the nonlinear term in NLSE is a sort of self-interacting quantity, wherein upper and lower signs, respectively, represent repulsive and attractive self-interactions. In view of this feature, Eq. (37) is also known as cubic Schrodinger equation.

However, before proceeding further, it may be pointed out that NLSE is a simplified version of the equations used by Ginzburg and Landau in 1950 in their study of the macroscopic theory of superconductivity, and by Ginzburg and Pitaevskii in 1958 in the theory of superfluidity. Furthermore, in 1964, Chiao et al and Talanov employed similar equation while investigating the phenomenon of self-focusing of optical beams and the conditions under which an electromagnetic beam can propagate without spreading in nonlinear media.

The soliton solution of Eq. (37), with + sign for the nonlinear term, determined by Zakharov and Shabat reads

$$u\left(x,t\right) = ae^{i\{\frac{v}{2}(x-vt)+bt\}}sech\{\frac{a(x-vt)}{\sqrt{2}}\}. \tag{38}$$

Here, the wave amplitude $a$, velocity $v$, and real constant $b$ are such that $a^2 = 2\left(b-\frac{v^2}{4}\right) > 0$. While writing this solution, the initial phase and the initial position appearing in the exponential and the sech terms have been assumed to be zero. The exponential term leads to an oscillatory component with amplitude dependent *sech* term as the envelope profile so that the resulting wave packet is a modulated one; Fig. 4. Such a solitary wave described by an envelope with an internal oscillation or pulsation, is called a breather. Sometimes, the terms envelope soliton and intrinsic localized modes are also used for this entity, particularly in nonlinear lattice dynamics. The $u\left(x,t\right)$ given by Eq. (38) represents a moving breather as it advances in space. In contrast, a breather solution has been obtained for the SGE that does not move and, hence, is referred to as a stationary breather.

In nonlinear optics, the breather solution that produces self-focusing of the carrier wave is known as bright soliton and the one giving self-defocusing is called the dark soliton.

It may be added that in addition to continuous NLSE, soliton solutions have also been found for the discrete nonlinear Schrödinger equation

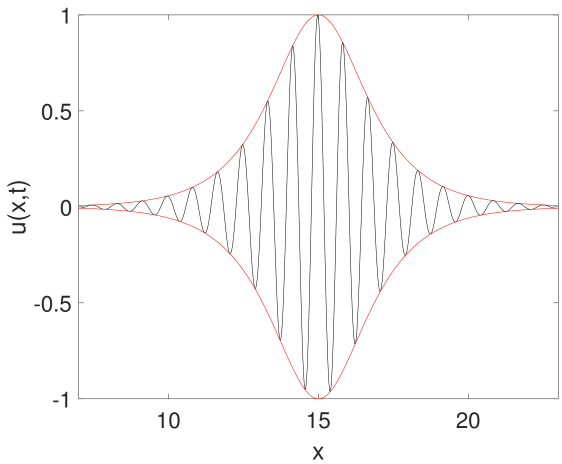


Figure 4: Profile of the breather solution given by real part of Eq. (38) for $a$ = 1.0, $v$ = 15.0, $t$ = 1.0, $b$ = 56.75. The black line depicting internal oscillations is confined within the red-line envelope soliton.

$$iu_{n,t} + u_{n+1} + u_{n-1} \pm |u_n|^2 u_n = 0, \tag{39}$$

and some of its generalizations leading to proper elaboration of many interesting properties of nonlinear lattice chains.

The models that are compliant with NLSE and its different variants have played an important role in the developments in nonlinear optics (light waves), soft-condensed matter physics particularly Bose-Einstein condensation (matter waves), fluid dynamics, plasma physics, etc. and continue to be valuable even now.

## 6 Some Applications of Solitons

The concept of solitons, including their different cousins, as sophisticated mathematical constructs to explore nonlinear phenomena has not only revolutionized research in mathematics as well as mathematical physics leading to advent of many new ideas and techniques, but has also been fruitfully exploited in developing numerous practical applications in different branches of science and engineering. Besides the three NLPDEs discussed in Sections 4 and 5, many other equations such as modified KdV equation, Benjamin-Ono equation, Boussinesq equation, Davydov's equations, etc. have been found to be of immense value. Most of different soliton bearing NLPDEs have been solved analytically as well as by numerical methods. In this section, we briefly describe some typical problems in various fields where solitons with their different manifestations have been employed, and we certainly do not claim exhaustiveness of the list. Also, the topics dealt with are being listed in alphabetical order.

1. Astrophysics and Cosmology

   (a) Electrostatic solitary waves have been experimentally observed in astrophysical plasmas such as the sun, the solar wind, lunar wake, the planetary magnetospheres, etc. Also, many theoretical models have been proposed to interpret the observed characteristics of these waves.

   (b) It has been shown that low dimensional black holes can be realized as solitons of the sine Gordon equation. Furthermore, it has been ferreted out that some field theoretic models for studying black holes also have soliton solutions indicating their intimate relationship.

   (c) The Great Red Spot of Jupiter (GRS), which is slightly oval and nearly 16,000 km wide, is an anticyclonic vortex that has persisted for hundreds of years of continuous observation despite the highly turbulent atmosphere on the planet. Two- and three- dimensional soliton models were put forward for this in 1980s, wherein the latter were found to be in better agreement with the then available data. However, some scientists believe that these models do not capture every minute detail of the GRS as soliton.

   (d) The solutions of some cosmological models with the cosmological constant have been found to exhibit the existence of solitary waves under specific conditions besides the travelling wave periodic solutions that resemble the gravito-static waves.

2. Biological Systems

   (a) The model developed for energy

transfer and energy coupling in hydrogen-bonded spines that span the length of protein $\alpha$-helices and stabilize it, encompasses the so called Davydov soliton. This soliton represents a state composed of an excitation of amide-I and its associated hydrogen-bond distortion. In fact, the relevant conjectures have been supported by different spectroscopic studies of proteins.

(b) The concept of Davydov soliton has also been used to describe a local conformational change of the deoxyribonucleic acid (DNA) $\alpha$-helix, which too has been confirmed by experiments.

(c) Generation of force in the sliding filament model of muscular contraction has also been attributed to Davydov solitons.

(d) The intrinsic localized modes, which arise from the anharmonicity of interatomic potentials, have been observed in proteins and identified as solitons localized in both space and time.

(e) The studies pertaining to electron / proton transport in $\alpha$ - helix sections of proteins, and the signal as well as energy propagation in lipid membranes have brought out the involvement of soliton-like mechanisms.

(f) Solitons obtained as solutions for the Peyrard–Bishop model (and its extended versions) put forward to understand the dynamics of DNA, explain important features like local opening (i.e., separation of double-stranded DNA into two single strands) and DNA transcription. A similar soliton-bearing model has also been developed to elucidate the long-distance charge transport in DNA molecule.

(g) The behaviour of many biopolymers has been explained in terms of breathers and this aspect too has been investigated experimentally.

(h) The concept of solitary waves has been recently used in neuroscience as an alternative to the earlier accepted ionic-hypothesis based Hodgkin–Huxley model to describe the propagation of signals along the excitable cells such as neurons and cardiac myocytes.

(i) Soliton-related mechanisms have been reported to play an important role in the eukaryotic multicellular movements during morphogenesis and development.

(j) It has been proposed that the blood pressure pulse is an outcome of a KdV soliton produced in the heart and its propagation in blood vessels.

3. Condensed Matter Physics

   (a) The phase-slip centres in the charge density wave condensate formed during phase transitions in which the electron density develops a small periodic distortion accompanied by a corresponding modulation of the ion equilibrium position, are solitons. These solitons are found in one-dimensional metals and organic conductors.

   (b) Solitons occur in structural phase transitions in quasi-one-dimensional ferroelectrics.

   (c) The flipping of spins in magnetic phase transitions in quasi-one-dimensional ferromagnets as well as antiferromagnets is associated with the kink solitons.

   (d) Solitons have been found to be instrumental in polymerization mechanism and creation of bond defects in polymers.

   (e) The phenomena of transport and existence of defects in two-dimensional Coulomb gases and two-dimensional spin systems (i.e., thin films) are understood in terms of solitons.

   (f) A domain wall or a Bloch wall in ferromagnets, ferrimagnets, ferroelectrics, etc. is an interface that separates magnetic or electric polarization domains of different types. These walls are exact solutions to SGE, NLSE, and their modifications and, hence, these have been identified as relevant solitons. These aspects have been experimentally verified by neutron scattering, NMR, and ESR studies in many materials.

   (g) Starting from the Frenkel-Kontorova model with on-site periodic potential, mentioned earlier in Section 5.1, the atomistic theories of crystal dislocations have been generalized to include physically more relevant non-sinusoidal and anharmonic interactions. Solitons, particularly kinks, have been found to play an important role in all these models and have been confirmed in experimental measurements.

   (h) Liquid crystals (which are used in display devices like televisions, computer monitors, laptop screens, calculators, etc,) are self-organized anisotropic fluids that are thermodynamically intermediate between the isotropic liquid and the crystalline solid, showing the fluidity of liquids and the order of crystals. Thus, these are mesophase entities. Being nonlinear materials, these have been widely used for creation and description of various types of solitons since 1968. The associated aspects have led to new applications of the liquid crystals.

(i) Solitons have been experimentally observed in thin superfluid $^4$He films (a few atomic layers thick) adsorbed on solid substrates (two-dimensional system) as well as bulk superfluid $^4$He (three-dimensional quantum material) and have been theoretically expounded using KdV equation and still better by employing phenomenological modeling based on time-dependent density functional theory.

(j) Different types of solitons have also been observed in ultracold superfluid $^3$He phases in the absence as well as presence of magnetic field (magnetic solitons). These have been explained theoretically using an NLSE like equation.

(k) An arrangement or device obtained by sandwiching a thin layer of a non-superconducting material (up to about 3 nm thick insulator or a few μm thick non-superconducting metal) between two layers of superconducting material, is known as a Josephson junction. It has a unique and important feature that a dc (supercurrent) can pass through the junction / barrier from one superconductor to the other even in the absence of an applied voltage and a sinusoidal ac current is generated when a fixed voltage is applied across it. The former is a consequence of quantum tunneling of Cooper pairs (pairs of electrons with opposite momenta and spins loosely bound at very low-temperatures due to electron-lattice interactions) across the nonconducting barrier and the latter makes it a nonlinear oscillator. These junctions find applications in quantum-mechanical circuits such as superconducting quantum interference devices (SQUIDs), superconducting qubits, and rapid signal flux quantum digital devices. The dynamics of the Josephson junction is reasonably well described by a perturbed SGE, which makes it a system for the study of solitons and phenomena associated with these. In fact, discrete breathers have been observed in arrays of Josephson junctions, and the solitons in the junctions which are much longer than characteristic Josephson penetration depth (which is of the order 1 – 1000 $\mu$m), are known as fluxons because they contain one quantum of magnetic flux ($h/2e$ = 2.07x10$^{-15}$ Wb; here $h$ is Planck's constant and $e$ is charge of an electron).

(l) The cumulation of a macroscopic

fraction of noninteracting identical boson particles (the entities having integer spin, which is actually an integer multiple of $\hbar = h/2\pi$, and is described by symmetric wavefunction) in the lowest energy or the ground state in a system under appropriate characteristic conditions of temperature, number density, etc. is known as Bose-Einstein (BE) condensation. It represents a phase transition to a state of matter in which a good number of constituents of the system suddenly coalesce into a single coherent quantum mechanical entity that can be described by a wavefunction on nearly macroscopic scale. The condensate appears as a sharp peak in both position and momentum space. The macroscopic dynamics of BE condensates near 0 K is generally modeled by a 3-dimensional version of NLSE with a term for the trap potential and is called the Gross–Pitaevskii equation. The solutions of this and other similar equations, lead to solitons of different types which have been observed experimentally as well. Besides, investigations on manipulating the properties of solitons in the BE condensates via nonlinearity management have also been carried out.

4. Engineering

   (a) Mathematical and computational studies of a variety of problems in theoretical aerodynamics have shown that in some situations, solitons can lead to chaotic motion.

   (b) The use of electrical components with nonlinear permittivity and permeability makes a transmission line to be nonlinear. In fact, such transmission lines (constructed with easily available components) constitute reasonably simple and low-cost experimental devices for investigating various aspects of nonlinear waves. These properly designed networks have been shown to produce (electrical) soliton pulses over a wide range of frequencies and find applications in wide band focusing and shaping of signals, and in instrumentation for microwave systems, in high-speed sampling oscilloscopes, and for data transmission in high-speed digital circuits, etc.

   (c) A structure such as a rod or a pile of plates with rectangular or circular cross-section made from some metal, polymeric materials, etc. that propagates elastic waves with minimal loss of energy by restricting their transmission along its length, is called an elastic or a solid waveguide. Such waveguides provided with piezoelectric transducers are used for measurement of strain, pressure, and temperature. In addition, these waveguides find applications in energy harvesting, vibration control, health monitoring, and wave steering for actuation. If the nonlinearity produced by properties of the constituent material and by the strain is compensated by the spatial dispersion caused by the finite transverse size of the waveguide, then longitudinal density solitary waves can be generated in it. These are generally described by Boussinesq-type NLPDE of elasto-dynamics and have been observed experimentally. These so-called strain or bulk solitons represent a powerful localized wave that can transport elastic energy over reasonably large distances with negligible losses.

(d) Granular crystals are nonlinear tailored metamaterials obtained from tight packing of macroscopic solid grains or particles like ball bearings made of a metal or an alloy or bits of polymers such as nylon, teflon, delrin, etc. (rather than atoms or molecules) that interact elastically. Like atoms in a crystal, the particles in a granular crystal can also be arranged in one-, two-, or three-dimensional lattices. The freedom to choose constituent particles with different masses, sizes, material properties, and geometries, and possibility to arrange these in a variety of configurations in a lattice, make the granular crystals highly tunable even in respect of the extent of nonlinearity. The dynamical description of these fabricated crystals brings out existence of traveling solitons as well as discrete breathers in these, which have been observed experimentally. These aspects, in turn, have made these engineered or manipulated materials useful as the shock-absorbers in armor and sports helmets; for sound-focusing devices, acoustic switches, acoustic logic elements; for mechanical vibrational energy harvesting systems; and for converting mechanical vibrations into electrical current that could drive small sensors or transmitters.

(e) The micro-electromechanical and nano-electromechanical systems, generally made from materials like carbon nanotubes and graphene, are artificial devices that combine electrical and mechanical processes at micro and nano scale, respectively. These find applications in automobiles, accelerometers, aerospace systems, sensors for environmental monitoring, defence systems, biomedical diagnostics, medical devices, signal processing, wireless communications, etc. Studies pertaining to dynamics of many such systems, particularly those comprising arrays of nonlinear oscillators, have established the existence of discrete breathers in these.

5. Hydrodynamics and Geophysics

   (a) The discovery of 'great wave of translation' by Russell in shallow water and its theoretical modeling by KdV equation, have motivated many scientists to study multifarious properties of shallow-water solitary waves in the laboratory. A variety of wave-tank experiments have been performed to investigate various aspects of these waves, including different types of collisions between solitons, and these continue to be of interest even now. Generally, the experimental results exhibit good agreement with relevant theoretical predictions. In addition, it is well recognized that various properties of shallow-water waves near the beaches are successfully explained by the KdV equation.

   (b) The surface waves observed in deep water have been identified as envelope solitary waves, whose theory was developed by Zakharov in terms of an NLPDE similar to the NLSE. These waves have also been investigated experimentally using large water tanks.

   (c) The soliton solutions of the KdV as well as the Benjamin-Ono equations have been used to describe internal gravity waves in the ocean, which are large amplitude waves travelling at low speed and originate from density differences caused by variations in temperature and saline concentration. These have been observed and painstakingly studied in many seas by oceanographers.

   (d) The seemingly spontaneous and extremely large rogue or monster waves too have been modelled as solitary waves.

   (e) It has been argued that strong velocity-dependence of amplitude of a solitary wave disturbance on the surface of water in an ocean created by an underwater earthquake, volcanic eruption, etc. makes its amplitude larger as the wave advances towards a beach. If the wave energy is quite high, then amplitude becomes so large that the wave breaks down into numerous waves of very large width (few hundred kilometers) and small amplitude (1 meter or so) as it reaches the beach. The catastrophe so created at the beach results in devastating tsunamis and hurricanes.

(f) Because of resemblance of some mountain ranges and layer distribution in some sedimentary rocks with envelope wave packets, NLPDEs based mathematical models have been developed to show that geo-solitons may have played an important role in their formation.

6. Nonlinear Optics

Nonlinear optics is the branch of optics that deals with the behaviour of light in the materials in which the electric polarization produced by the electric field of the light passing through it varies as higher powers of the electric field strength, i.e. nonlinearly, particularly when the light intensity is high. When a highly intense beam of laser radiation propagates through a material like silica-based glass, lithium niobate, etc., additional phase shift (called self-phase modulation) is introduced due to intensity dependent refractive index (the Kerr effect). This nonlinear phase shift in the pulse leads to its shrinkage in contrast with spreading produced by dispersion. If these two opposing effects cancel each other, we get temporal optical solitons. Besides temporal optical solitons, spatial optical solitons have also been found to exist in many nonlinear media. When an intense light beam passes through such bulk materials along, say, $x$ – direction, it may undergo diffraction along the two transverse directions. If the broadening produced by diffraction is counterbalanced by the narrowing caused by the nonlinearity associated with intensity dependent refractive index, spatial optical solitons are obtained. In addition to the temporal and spatial solitons, spatiotemporal optical solitons (where both the diffraction and dispersion effects are simultaneously compensated by nonlinearity) have also been created in some nonlinear optical materials. In a nutshell, an optical soliton refers to a situation where light beam or pulse (self-trapped in time or space or both) travels through a nonlinear optical material without any change in its profile and velocity. These solitons are mathematically described by NLSE (continuous as well as discrete) and are found in photonic crystal fibres, photorefractive materials, photopolymers, etc.

(a) The idea of temporal soliton transmission in glass fibre waveguide (or an optical fibre) was put forth by Hasegawa and Tappert in 1973 on the basis of theoretical and numerical calculations, and its experimental observation in silica-glass fibre was reported by Mollenauer et al in 1980. When laser pulses are used for communication employing optical fibre, the solitons involved are sometimes referred to as fibre-solitons. Presently, it is possible to propagate solitons without degradation over thousands of kilometers. Such a communication has zero loss and no dispersion, which explains the focus of a great research effort to understand the dynamics of soliton transmission in optical fibres. Furthermore, it brings out the importance of optical fibre communication in information technology and in the long-distance, high bandwidth communication – the well-known internet and the world-wide web.

(b) The spatial solitons in photorefractive polymers make these highly efficient optical elements for transmission of data and for controlling coherent radiation in various electro-optical and optical communication devices.

(c) Ultra-short pulse solitons are being used in the field of optical spectroscopy and medicine.

(d) Optical solitons in birefringent optical fibres are used for optical switching.

(e) The fabrication of materials with extremely strong nonlinear effect has made it possible to create optical solitons even with very low laser powers. These find applications in optical information storage of large amount of data, all-optical switches, and significantly faster optical systems than any known electronic devices. These concepts form essential basis of the possible optical digital computer system or the photonic computer with solitons as bits.

(f) Light or optical bullets which are three-dimensional localized pulses of electromagnetic energy and have been observed in arrangements like array of silica glass waveguides, sapphire samples, plasmas, etc., are examples of spatiotemporal solitons. However, these lose energy during interactions / collisions implying that these are not solitons in the strict sense of the term.

7. Nuclear Physics

(a) Topological solitons have been found to describe reasonably well some properties of nuclei, including prediction of binding energies to the correct nuclear physics level. This aspect has been found to have substantial impact on the studies pertaining to nuclear matter in neutron stars and in nuclear fusion.

(b) Nontopological soliton models based on simple phenomenological field theories have been used to incorporate the quark structure of hadrons in nuclear physics.

(c) It has been shown that the velocity dependent terms in the nucleon-nucleon potential lead to formation of solitons in nuclear matter, which play role in nuclear multi-fragmentation reactions.

8. Plasma Physics

Plasma physics deals with the study of matter consisting of a large number of charged particles – ions and / or electrons. The presence of inter-particle coulomb interaction makes plasmas a nonlinear system and, as such, these offer a good testing ground for the study of solitons.

(a) The KdV and some other similar NLPDEs have been used to describe the local charge density reflecting the local departure of the charge from neutrality in the plasmas, and, thus, establishing the presence of travelling solitons in these.

(b) Ion-acoustic solitary waves have been theoretically and experimentally studied in magnetized plasma.

(c) The Alfvén waves, observed in plasmas on the earth and in the space, are low-frequency travelling oscillations of the ions caused by the interaction of the magnetic fields and electric currents within the plasma. These magneto-hydrodynamic waves were among the first to be modeled using idea of solitons.

(d) Dusty plasmas, which contain small suspended particles, have been modeled using nonlinear oscillator chains, showing the existence of discrete breathers in these.

(e) It is well known that space debris objects, whose number in earth's orbit is estimated to be few hundred million, pose immense threat to the earth-orbiting satellites. Also, these objects get electrically charged due to their exposure to the ionospheric plasma environment. Some recent analytical, computational, and experimental investigations have shown that charged objects moving with high speed through a plasma lead to generation of plasma density solitons. Accordingly, depending on its size, charge and velocity, debris object will produce solitons, which can be detected by fixing simple instruments on the spacecraft.

9. Quantum Mechanics, Elementary Particle Physics, and Field Theory

(a) With a view to develop a classical interpretation of quantum mechanics, Bohm and others treated quantum processes as stochastic processes. This approach was subsequently used to derive a nonlinear relativistic Klein-Gordon equation yielding soliton solutions, which follow the average de Broglie-Bohm trajectories analogous to the linear solutions of the Schrödinger and the Klein-Gordon equations. These ideas have been extended to show that even photon can be represented as a soliton. A relationship between the electromagnetic amplitude of this soliton and photon energy or frequency has been established. Also, it has been proved that the concept of photon-soliton is in conformity with the familiar interactions in the photoelectric and Compton effects.

(b) Recall that solitons are confinement of energy of the wave-field, propagate without change in shape, collide like particles, and a soliton-antisoliton pair may get annihilated. In view of these facts, it was conjectured that if an appropriate system of nonlinear field equations admits soliton-solutions then these may represent elementary particles. As such, 'bags' and 'lumps' in quantum fields are described in terms of solitons. However, many of these issues are still being debated [4].

(c) The instanton solutions of Yang-Mills field equations used for unifying electromagnetic and weak forces are soliton-like because these are localized in space as well as time.

(d) In order to explain the stability of protons, neutrons, and mesons, Skyrme (1961) developed a model in which these elementary particles could be treated as topological defect solitons in a quantum field. This stable field configuration with special topological properties came to be known as skyrmion. However, this idea did not find much ground in particle physics even though it accounted for some low-energy properties of the nuclear particles. Interestingly, skyrmion-like topologies have been found to exist in many condensed matter systems such as some liquid crystal phases, BE condensates, quantum Hall systems, and helimagnetic materials in which neighbouring magnetic moments arrange themselves in a helical or spiral pattern. In the last category of materials exemplified by FeGe, Tb, Dy, etc. these form domains as small as 1 nm and involve extremely low energy. These features make magnetic skyrmions a good option for developing very efficient memorystorage and other spintronics devices. In fact, the activities in this direction constitute the emerging field called skyrmionics.

(e) The Einstein field equation, which constitutes the backbone of the general theory of relativity, describes gravity to be a consequence of spacetime being curved by both mass and energy. It is, in fact, a set of ten NLPDEs in four independent space and time variables, expressed as a tensor equation. Nonlinearity of the equation leads to a solution, which has soliton characteristics (confined to a finite region of spacetime and has a finite energy) and is called the gravitational soliton. It can be separated into two kinds - a soliton of the vacuum Einstein field equation and a soliton of the Einstein–Maxwell equations. Even black holes, the main sources of gravitational radiation, are two-soliton solutions of Einstein's equations in vacuum.

## 7 Epilogue

It is indeed very interesting to note that solitons occur over a wide range of scales. On one side, these have been found to be extremely useful in understanding various phenomena at the nuclear and atomic level, though their experimental manifestation is not that straightforward. On the other hand, these have been extensively observed and manipulated in the macroworld. The linear dimension of nuclear solitons is few femtometer ($10^{-15}$ m), of the optical solitons is few nm ($10^{-9}$ m), of the solitons observed on the surfaces of water bodies is few cm to few meters or even few hundred kilometers, and of the GRS solitons is thousands of kilometers. Solitons associated with BE condensates are observed at ultra-low temperatures of $10^{-7}$ K or so, the hydrodynamic solitons occur at around 300 K, the temperature over the GRS is about 1600 K, and core temperature of the sun is about $10^{7}$ K.

The distinctive behaviour of Josephson junctions including magnetic flux quantization, superfluidity of helium, and Bose-Einstein condensation (discussed under condensed matter physics in Section 6) are manifestations of quantum effects at macroscopic level. All of these have their origin in the collective coherent behaviour of constituent quantum particles with nonlinear interactions, which balance the dispersive effect of kinetic energy. As mentioned earlier also different species of solitons (which are coherent structures created by perfect balance between effects of nonlinearity and dispersion) have not only been predicted in these systems using relevant NLPDEs (like NLSE, etc.) but have also been observed experimentally. Thus, the above-mentioned systems are nonlinear quantum phenomena where the Hamiltonian is a nonlinear function of the wavefunction of the microscopic entities involved. It has been argued (see, e.g. [4]) that nonlinear quantum theory based on NLPDEs with solitons as integral part be developed in proper perspective to describe such systems and to investigate related features in detail. It can be said without any exaggeration that this development will act as stimulant for a new surge of soliton-oriented activities in condensed matter physics, polymer science, and biophysics.

Lastly, to conclude the article, we quote Kasman [9] "solitons have become (*vital*) tools of scientists and engineers for understanding the universe".

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