Tensors: Significance and Features

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Abstract

At the first face-off with Tensors, they appear to be scary and formidable. Even Einstein had to struggle a lot to master it. But once well versed with the subject Einstein used it aptly in the formulation of his masterpiece General Theory of Relativity (GTR), whose fame in due time made the subject of tensors synonymous with GTR in Physics. Earlier, however, with their little use in other areas of physics, students not opting for GTR could turn a blind eye to them. But since GTR much water has flown down the tensor pipeline, and the subject has evolved a lot with numerous applications not only in Physics and Mathematics but in fields as varied as Computer Science, Chemistry, Geology, Statistics, Medicine, Engineering, etc. In Physics, it is now regarded as an indispensable tool for the description of all the four fundamental interactions. Further, an operation on tensors called tensor product is a pre-requisite for the description of quantum states when two or more quantum systems get together, and also, the entangled states in their joint vector space need tensors for their expression. Very significantly, the fifth aspect of tensors, apart from its ability to represent invariance, anisotropicity, many quantum systems states and entanglement, is the capacity for large data storage, which is an artifact of the fact that high-rank tensors can be effectively represented by multi-dimensional hyper matrices. This feature of tensors has come to great advantage in Computer Science, where it is utilised for organising or storing large data and data mining with bearing on machine learning, deep learning, tensor imaging, face recognition, computer vision, etc. This article is a modest attempt (as the subject is deep and profound and cannot be justified in an article of over a dozen pages) to make accessible the features of tensors and their significance to the undergraduate students. The goal here is not to provide the students a working knowledge of tensors but to entice them by showing them the wonderful world of tensors so that they learn it on their own.

1 Tensors: Etymology, Origin and Development

It appears that the English word *Tensor*, which owes its origin to the Latin word *Tensus* meaning **Tension**, has an influence on its import. *Box 1* quotes a few tensor anecdotes that testify to tensor's reputation or notoriety for being daunting. Incidentally, the first tensor used in physics has a close connotation with its meaning, as it is none other than the famous *stress* tensor. A tensorial expression, represented by some symbols adorned with multiple indices in subscript and superscript fashion, projects a frightening sight to anyone who wants to comprehend its meaning. In case the symbol appears pleasing to some gutsy person and emboldens him/her to read the modern highly abstract definition *'A tensor is a binary covariant functor [1] that represents a solution for a co-universal mapping problem on the category [2] of vector spaces over a field,'* will certainly spin his/her head. This no doubt looks quite intractable at the first sight. But a little familiarity with tensors makes one regardful of how important a tool it is to express equations of physics, notwithstanding the other important applications which the tensors lend themselves to.

The word *'tensor'* was introduced by William Rowan Hamilton (1805–1865), initially to describe something different from what is now meant by a tensor (namely, the norm operation in a certain type of algebraic system now known as *Clifford algebra*). The contemporary usage was introduced by Woldemar Voigt around 1898. The concept of tensors has its origin in the development of differential geometry by mathematical stalwarts no less than Carl Friedrich Gauss (1777-1855) and Bernhard Riemann (1826-1866). Later, Elwin Bruno Christoffel's (1829-1900) work in differential geometry, particularly the connection formulae obtained by him to express covariant derivatives, paved the way for tensor calculus. Gregorio Ricci-Curbastro (1853-1925) and his Student Tullio Levi-Civita (1873–1941) generalized Christoffel's ideas and developed them further to institute the concept of tensors and absolute differential calculus. *Figure 1* contains photographs with names printed below each of a few of the progenitors of the subject of tensors. The absolute differential calculus, later known as tensor calculus, forms the mathematical basis of the general theory of relativity, which popularized the subject by leaps and bounds. From 1920 onwards, tensor concepts pro-

Figure 1: Few Tensor Progenitors Photographs with their names.

gressed to newer, more abstract areas that is from differential geometry to topological algebra, Topological algebra [3] and more recently, to category theory. It won't be an exaggeration to say that the study of tensors is a study in the progress of mathematical thought. The stated tensor definitions in *Box 2* allude to this evolution of mathematical ideas.

- General relativity is formulated completely in the language of tensors. Einstein had learned about them with great difficulty from the geometer Marcel Grossman.
- Levi-Civita during 1915-17 initiated a correspondence with Einstein to correct mistakes Einstein had made in his use of tensor analysis.
- Albert Einstein in a letter to Tullio Levi-Civita wrote: I admire the elegance of your method of computation: it must be nice to ride through these fields upon the horse of true mathematics while the like of us have to make our way laboriously on foot.
- When an interviewer questioned, Professor Eddington, is it true that only three people in the world understand Einstein's theory? Eddington retorted: Who is the third?
- A professor at Stanford once said, 'If you really want to impress your friend and confound your enemies, you can invoke tensor products... people run in terror from the $\otimes$ symbol.

BOX 1. Tensor Anecdotes.

2 Tensor Applications

These days abundance of literature on tensors being copiously produced by mathematicians, physicists, computer scientists, statisticians and engineers as well as experts in other scientific fields signify to the importance that tensors hold in Science and Engineering. As mentioned above, few decades before tensors were almost synonymous with General Relativity- except for a minor use in all other branches of Physics. The realization that gauge fields are geometrical objects has made the geometrical (coordinate-independent) aspect of tensors become more and more significant in the study of all interactions as all fundamental interactions including gravity are deemed to be different manifestations of the same super force.

In recent decades, relativistic quantum field theories, gauge field theories, and various unified field theories have all used tensor algebra analysis exhaustively. Also tensor products naturally arise in quantum mechanics as a description of many particle state space because they can take into account the superposition aspect of quantum states when separate quantum systems are brought together. Further the fast burgeoning field of quantum computation hinges on the concept of entangled states which need tensors for their formulation. In mathematics tensors are used in Differential Geometry, Differential Equations, Spectral Theory, Continuum Mechanics, Fluid Dynamics, Multilinear systems in Numerical Algebra, Tensor complementarity problems, Optimization, etc.

One of the most important applications of tensors is to tensor decomposition that is presently used for applications in numerous varying fields. Though tensor decomposition methods have appeared as early as 1927, but they remained unused in computer science field as late as the end of 20th century. An early use of tensor decomposition was sought in the area of psychometrics which deals with intelligence evaluation and other personality characteristics. But in the last two decades, a growing computing capacity and an increasing familiarity with multilinear algebra have led tensors to emerge in a big way in the earlier untouched areas of statistics, data science and machine learning. In data science, real data are often in high dimensions with multiple aspects and tensors provide elegant theory and algorithms for web data mining, face recognition softwares, higher order diffusion tensor imaging in medical imaging, psychometrics, chemometrics, neuroscience, graph analysis, fluorescence spectroscopy, geophysics, etc. In each case, data is compiled into a multi-way array or a hyper matrix and the essential features of the data are isolated by decomposing the corresponding tensor into sum of rank one tensors.

A. Poorman's Definition; Rather Impression
- Tensor is nothing but index gymnastics played with certain rules.

B. Heuristic Definition
- Tensor is what that transforms like a tensor.

C. Canonical Definitions
Tensor(s) is(are);
- a generalisation of vectors & co-vectors.
- an invariant abstract or geometric object with a magnitude and several directions.
- a mathematical entity that transforms according to certain transformation laws.
- a multi-dimensional array of numbers or a n-dimensional generalization of a matrix.
- are just vectors in a special vector space or an element of tensor product space.
- a multi-linear operator that maps vectors and co-vectors to real numbers.
- a binary covariant functor.

BOX 2. Qualitative definitions of Tensors.

3 Approaching Tensors

In physics any quantity that has both magnitude and direction is a vector. Displacement, velocity, acceleration, and force are few examples of mechanical vectors. In three dimensional Cartesian space, a vector is represented by its x, y, z components. If we multiply this vector by a scalar quantity, all the three components of the vector scale up proportionately or, in other words, the vector changes its magnitude without changing its direction.

What if we want to create a new vector with a different magnitude as well as direction than the initial vector? Multiplication by a scalar only changes the magnitude. Taking the inner product with another vector turns it into a scalar, and in this way, the direction too is lost. Forming the cross product with another vector, though it changes the direction, always does so in the normal direction. So, for changing direction in an arbitrary way, we either take the *outer product* of a vector with another vector and obtain a second-rank tensor having a magnitude and two directions, or multiply the initial vector by a new mathematical entity called a *tensor* and obtain a tensor of higher rank, having a magnitude and multiple directions. *Table 1* presents the resultant quantities obtained from various multiplicative products of scalars and vectors, along with their examples in physics in three-dimensional Cartesian space.

A physical example of a tensor of rank two is force acting on a plane surface area. In this case, both the magnitude and direction of the force, and the size and orientation of the area, will determine the total effect. The size of the area and its orientation can be represented uniquely by a vector whose magnitude is proportional to the area size and whose direction is normal to the surface. Therefore, the effect of the force upon the surface depends on two vectors, the force vector and the area vector, and hence is described by a tensor of second rank. Second-rank tensors appear in physics when physical quantities exhibit anisotropic behaviour in the system, often in a "stimulus-response" mode, as discussed in the next section. In general, a second-order tensor, which takes in a vector of some magnitude and direction, returns another vector of a different magnitude and direction. If we take into consideration the components of force, each of the components acting on each component of the area vector, then there are nine terms altogether, which can succinctly be arranged in matrix of order 3 representing the total stress. So tensors can thus be represented by arrays, and manipulated in a manner reminiscent of matrix manipulation. The *Figure 2* shows tensors of zero, first, second and third rank as dimensional arrays or matrices. The single and multi-dimensional different stress components and *Figure 3* exhibits the distress tensor of a point in 3D space.

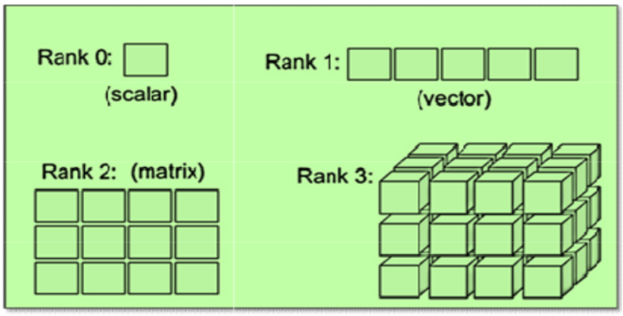

Figure 2: Tensors as multi-dimensional (Hypermatrix) array of numbers.

3.1 Tensor Definition

Tensors have been defined in several equivalent ways. These definitions can be broadly classified into two main types. The first type is traditional and defines tensors using coordinate transformation properties of components of tensors, whereas the second type is more modern and abstract and defines tensors in their component free formulation. We will briefly discuss only the first definition, due to constraints of the article size, but encourage the reader to learn about the second type in the suggested readings.

As remarked, tensors are usually intro-

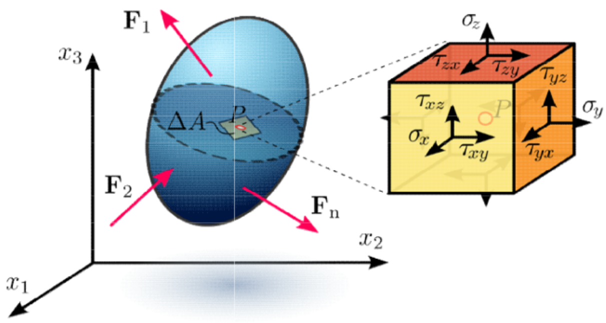

Figure 3: Stress tensor components in 3D space.

duced in terms of tensor components transformation rule. A tensor consists of tensor components and an underlying basis vectors of the coordinate system in which it is referred to. For the ease of understanding we consider the simplest case of orthonormal Cartesian coordinate system in Euclidean space. In this system the basis unit vectors are constant and so it suffices to give the components of a tensor. Another thing is that the transformations between Euclidean bases are always orthogonal. An orthogonal transformation of tensors from one Euclidean space to another preserves the length, and also makes no distinction between covariant [4] and contravariant [5] tensors. If $T(x) = Mx$ is an orthogonal transformation, we say that M is an orthogonal matrix. And from matrix theory we know that a matrix is orthogonal iff its inverse and transpose are the same, i.e., $M^{-1} = M^{\mathrm{T}}$.

We shall now examine the behaviour of a low order tensor of rank one that is a vector if we move from a two dimensional (2D) Cartesian coordinate system S to another 2D Cartesian system S′. The case of transformation rule of scalars which are tensors of the lowest rank is trivial because scalars are independent of the choice of coordinate system and does not require basis vector for their description. The 2D Cartesian S′ coordinate system in consideration is rotated by an angle ϕ with respect to the S system, as shown in the *Figure 4*. Let $\mathbf{E}$ be an electric field vector lying on a 2D plane, the vector making an angle θ with the x-axis in the S system. Then the components of $\mathbf{E}$ in the S system are $E_x = |\mathbf{E}|\cos\theta$ and $E_y = |\mathbf{E}|\sin\theta$. The coordinates of the electric field vector in the rotated system S′ will be,

$$E'_x = |\mathbf{E}|\cos(\theta-\phi) \tag{1}$$
$$= |\mathbf{E}|\cos\theta\cos\phi + |\mathbf{E}|\sin\theta\sin\phi \tag{2}$$
$$E'_y = |\mathbf{E}|\sin(\theta-\phi) \tag{3}$$
$$= |\mathbf{E}|\sin\theta\cos\phi - |\mathbf{E}|\cos\theta\sin\phi \tag{4}$$

Using $E_x = |\mathbf{E}|\cos\theta$ and $E_y = |\mathbf{E}|\sin\theta$, the above Eqs. (1) and (2) become,

$$E'_x = E_x\cos\phi + E_y\sin\phi \tag{5}$$
$$E'_y = E_y\cos\phi - E_x\sin\phi \tag{6}$$

These transformation equations can be written in matrix form as,

$$\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

The matrix form of Eq. (5) can simply be written as $\mathbf{E}' = M\mathbf{E}$, where

$$M = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

is a 2×2 matrix and $\mathbf{E}'$ and $\mathbf{E}$ are 2×1 column matrices. Since the elements in the matrix are identified by their row and column positions, the transformation Eqs. (3) and (4) can also be put as

$$E'_i = \sum_j a_{ij} E_j \tag{7}$$

Where the indices i and j take the variables x and y, and the direction cosine coefficients a_{ij} are: $a_{xx} = \cos\phi$, $a_{xy} = \sin\phi$, $a_{yx} =$

Product Name	Initial Entities	Initial Direction	Resultant quantity	Final Directions	Example in 3D Euclidean Space
Scalar	Scalar with Scalar	Zero, Zero	Scalar	Zero	Energy=Boltzmann constant times temperature ($E=K_BT$) (Note in Minkowski 4D space, energy is the Zeroth component of the momentum four vector, so it's not a scalar.)
Scalar	Scalar with Vector	Zero, One	Vector	One	Force= mass times acceleration (F=ma)
Inner	Vector with Vector	One, One	Scalar	Zero	Power=Force times Velocity (P=F.v)
Vector	Vector with Vector	One, One	Vector in Normal direction	One	Angular Momentum*= distance times momentum ($\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$) (*Not in relativity)
Dyad/ Outer/ Tensor	Vector with Vector	One, One	Tensor of second rank	Two	Moment of inertia $\mathbf{I} = \sum m_i \left({r_i}^2 \mathbf{1} - \boldsymbol{r_i}\boldsymbol{r_i}\right)$ Where $\mathbf{1} = \boldsymbol{ii} + \boldsymbol{jj} + \boldsymbol{kk}$ is a unit dyadic.

TABLE 1. Resultant tensor quantity from various multiplicative products of scalars and vectors and their examples in Physics.

$-\sin\phi, \quad a_{yy} = \cos\phi.$

Taking partial differential of Eq. (7) with respect to each of the components, and putting them into a matrix yields the following:

$$\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \begin{bmatrix} a_{xx} = \frac{\partial E'_x}{\partial E_x} & a_{xy} = \frac{\partial E'_x}{\partial E_y} \\ a_{yx} = \frac{\partial E'_y}{\partial E_x} & a_{yy} = \frac{\partial E'_y}{\partial E_y} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

Thus, the vector (tensor of rank 1) transformation rule can also be succinctly cast as:

$$E'_i = \sum_j \frac{\partial E'_i}{\partial E_j} E_j \tag{8}$$

Now, by just noting that the transformations in Euclidean space are orthogonal, we can write the inverse transformation equation by inverting the matrix M, which amounts to just transposing the row elements with column elements, that is, replacing a_{ij} with a_{ji}.

$$M^{-1} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

One can easily check that $|M| = |M^{-1}| = 1$ and that $|M||M^{-1}| = I$, meaning that the tensor remains invariant under rotation transformation. Using this fact, the inverse transformation equations $\mathbf{E} = M^{-1}\mathbf{E}'$ can be

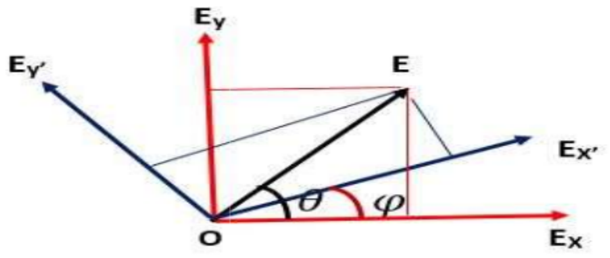

Figure 4: Electric field vector E and their components in 2D Cartesian coordinate system .

written as:

$$E_i = \sum_j a_{ji} E'_j = \sum_j \frac{\partial E_i}{\partial E'_j} E'_j \tag{9}$$

Thus, we infer that a Cartesian vector is an invariant physical quantity that transforms from the S coordinate system to the S' coordinate system according to the Cartesian tensor transformation law given by Eq. (9), and by Eq. (11) for vice-versa. If we generalise this definition, then we have the following definition of a rank n tensor:
A Cartesian tensor of rank n is a set of N^n quantities $T_{ij...m}$, which transform under rotations according to the rule:

$$T_{ij...m} = \sum_p \sum_q \cdots \sum_t T_{pq...t} a_{ip} a_{jq} \ldots a_{mt} \tag{10}$$

where, $a_{ip} a_{jq} \ldots a_{mt}$ are the cosines of the angles between the new and old coordinates.

4 Tensor Features

Though the ability to express invariance[6] is a fundamental property of tensors, besides this main property, four other innate potentialities possessed by tensors come in handy to express various aspects of physical reality in science. These aspects/features or characteristics are namely: Anisotropicity, Many-particle quantum states, Entanglement, and Big data storage capacity.

4.1 Invariance (Covariance of Physical Laws)

The main characteristic of a tensor is that its representations in different coordinate systems depend only on the relative orientations and scales of the coordinate axes at that point, and not on the absolute values of the coordinates. Tensors serve to seclude the intrinsic geometric and physical properties from the coordinate dependent ones. So if two tensors of the same type are equal in one coordinate system, then they are equal in all coordinate systems. Therefore it can be said that the central principle of tensor analysis amounts to the simple fact that tensors remain invariant with coordinate transformations. This implies that equations written in tensor form are valid in any coordinate system as tensor equations look the same in all coordinate systems. This is why the absolute position vector pointing from the origin to a particular object in space is not a tensor because the components of its representation depend on the absolute values of the coordinates.
The physical reality encoded in the laws of physics is universal that is independent of reference frames under appropriate symmetry transformations. So this means it depends on what laws one is talking about,

as for instance Newton's laws are invariant with respect to the Galilean transformations and Standard model is invariant with respect to Lorentz transformations. In both these theories there is a preferred set of frames called inertial frames. The theories or the laws are invariant only with respect to which inertial frame one is using. In contrast, general relativity is invariant with respect to general coordinate transformations. And as remarked above, the main characteristics of objects called tensors is that they remain invariant under certain coordinate transformations. So it should be clear that invariance of tensors is subject to transformation rules. One should first talk about the transformations under which one is asking for invariance. Only then, logically speaking, can one talk about tensors. The same object could have different transformation properties with respect to different transformations. For example, Higgs boson (before electroweak symmetry breaking) is an SU(2) doublet while Lorentz scalar. So, as a tensor it will have only one SU(2) index and no Lorentz index.

This entails that if laws of physics are expressed using tensors they become form invariant under appropriate transformations and hence tensors provide the best means to objectively represent the physical reality independent of coordinate systems or observers. In the language of physics if the equations of physics possess the same form in different coordinate systems they are said to be covariant, though the word covariant

Tensors equations are covariant (take the same form) in all coordinate systems

As an illustration of this fact consider a tensor equation of the form as given below:

$$U^i_j = \rho W^i_j \qquad (1)$$

where U^i_j and W^i_j are mixed tensors of rank 2 and ρ is some constant. Under a coordinate transformation U^i_j and W^i_j transform as:

$$U^\alpha_\beta = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial x^j}{\partial y^\beta} U^i_j \qquad (2)$$

$$W^\alpha_\beta = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial x^j}{\partial y^\beta} W^i_j \qquad (3)$$

And the equation (1) transforms as:

$$\frac{\partial y^\alpha}{\partial x^i}\frac{\partial x^j}{\partial y^\beta} U^i_j = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial x^j}{\partial y^\beta} \rho W^i_j \qquad (4)$$

With the help of equations (2) and (3), equation (3) reduces to;

$$U^\alpha_\beta = \rho W^\alpha_\beta \qquad (5)$$

Comparing the old coordinate equation (1) with the new coordinate equation (5) we observe that they wear the same form.

BOX 3. Illustration of the invariance of Tensor equation.

in tensor analysis has different meaning too. The *Box 3* illustrates with an example the covariant nature of a tensor equation. This demand of covariance, that all physical laws should be invariant under transformation between inertial systems led Einstein to the formulation of his theory of special relativity. Also, the need that the Maxwell's equations should be invariant under transformations, and the failure of Galilean transformations to do it, led to the Lorentz transformations. In *Table 2* we present few familiar equations (Newton's 2^{nd} law, Maxwell's equations and Dirac equation) in their usual scalar/vector form and the same in tensorial form.

4.2 Anisotropy (in properties of Material, Fields and Manifolds)

When materials are subjected to some stimulus like mechanical force, electric field, magnetic field, temperature field, etc., they subsequently respond, which respectively may be reflected in some property change such as elongation/deformation, electric current or polarization in dielectrics, magnetization, heat flow, etc. But these responses may also be associated with some other cause/stimulus. For instance, the application of pressure may lead to the polarization of the crystal (Piezo-electric effect) or the influence of a magnetic field may lead to strain in the material (magnetostriction), or the presence of temperature difference can cause electrical potential difference (pyroelectricity). In all such cases, a material/physical property connects the stimulus to the response, like, $P = p \cdot \Delta T, \quad M = \chi \cdot H, \quad \sigma = c \cdot \varepsilon$, etc., where the symbols are defined in *Table 3*. These material properties can be measured in experiments or can be calculated from more fundamental properties.

In reality, the stimulus and response are usually direction dependent or *anisotropic* and hence are tensors, and therefore the material properties are also tensors of some rank. The equations mentioned above take the following look in tensorial notation:

$$P_\alpha = p_\alpha \Delta T, \quad M_\alpha = \chi_{\alpha\beta} H_\beta, \quad \sigma_{\alpha\beta} = c_{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta}$$

The physical property connecting a stimulus of rank-m to a response of rank-n will be of rank $(m+n)$. In the first equation listed, the stimulus ΔT is of rank zero and the response P_α is of rank one; hence the property p_α is of rank one, in the second equation, the stimulus H_β is of rank one and the response M_α is also of rank one; hence the property tensor $\chi_{\alpha\beta}$ is of rank two, in the last equation, the stimulus $\varepsilon_{\alpha\beta}$ is also of rank two and the response $\sigma_{\alpha\beta}$ is also of rank two; hence the property tensor $c_{\alpha\beta\gamma\delta}$ is of rank four. Few of These material properties in tensorial and usual representations, along with their corresponding stimulus, response, and ranks, are tabulated in *Table 3*.

As mentioned above, apart from the material properties, tensors are also used to describe fields and manifolds. Similar to property tensors, these tensors can be of various ranks. For example, the temperature field $T(x, y, z)$ is a scalar field, where each point in space is described by one number at that point. Hence, scalar fields are tensor fields of rank zero. On the other hand, electric and magnetic fields are vector fields or tensor fields of rank one, and their specification requires three numbers at each point in three-dimensional space. These three numbers are the components along the coordinate axes and give the direction and magnitude of the vector. The electromagnetic field tensor, introduced after the four-dimensional tensor formulation of special relativity in Minkowski space-time, is a second-rank tensor, and the electric and magnetic fields can be obtained from the components of the electromagnetic tensor.

Another example of a tensor field is the *Riemann curvature* of a manifold. A *manifold* is a topological space that locally resembles Euclidean space near each point. When distances and angles can be measured on the manifold, then it is called *Riemannian*.

Equation Name	Scalar/Vector Equation	Tensor Equation
Newton's 2nd Law	$m\frac{d\vec{v}}{dt}$	$m\frac{d\mu^{\mu}(\tau)}{d\tau}=f^{\mu}$
Maxwell Equations	$\nabla.\vec{B}=0$	$\partial_{\mu}(\frac{1}{2}\epsilon^{\mu\lambda\alpha\beta})F_{\alpha\beta}=0$
	$\nabla\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$	
	$\nabla.\vec{E}=4\pi\rho$	$\partial_{\beta}F^{\alpha\beta}=\frac{4\pi}{c}j^{\alpha}$
	$\nabla\times\vec{B}-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}=\frac{4\pi}{c}\vec{j}$	
Dirac Equation	$\left(\beta mc^2 + c\sum_{n=1}^{3}\alpha_n p_n\right)\psi=0$	$(i\hbar\gamma^{\mu}\partial_{\mu}-mc)\psi=0$

TABLE 2. Few familiar undergraduate scalar/vector equations in Physics in their tensorial attire.

More distinctly, a *Riemannian manifold* is a differentiable manifold in which each tangent space is equipped with an inner product in a manner which varies smoothly from point to point. In tensor analysis, the *Riemann curvature tensor* is the most common way to express the curvature of Riemannian manifolds. It assigns a tensor to each point of a Riemannian manifold (i.e., it is a tensor field) that measures the extent to which the metric tensor is not locally isometric to that of Euclidean space. It is the sophistication or elegance of tensor analysis that it is able to capture the invariance aspect as well as the spatial peculiarities in one go. This, at once, can be seen from the general formula for the invariant line element in any space, $ds^2 = g_{ij}\,dx^i\,dx^j$, where g_{ij} is the metric tensor encoding the properties of the space.

4.3 Many System Quantum States

In quantum mechanics, a tensor product is used to describe a system that is made up of multiple quantum subsystems. The simple reason that tensor product is required to build the joint space is because the dimension of joint vector space of two separate quantum systems magnifies multiplicatively and not additively, and is equa to the product of dimensions of the two separate system vector spaces, i.e., $\dim(V \otimes V) = (\dim V)(\dim V)$. If V is the vector space of one system and V is the vector space of another system then the quantum state of both the systems is $V \otimes V$, where the symbol represents tensor product.

Now it is well known in tensor analysis that the rank of a tensor can be increased through the outer or tensor product. If we take Cartesian product of the two vector spaces $V \times V$ then the resultant dimension is just the direct sum of V and V i.e., $\dim(V \times V) = \dim V + \dim V$, because the vectors are ordered pairs of vectors $(V, V) \in V \times V$. The cartesian product space $V \times V$ is a space whose states are the states of system V or system V or both, whereas $V \otimes V$ is the vector space whose basic states are pairs of states, one from V and one from V. So the Cartesian product cannot account for a large Hilbert space constructed from the smaller sub Hilbert spaces but this large Hilbert space is accounted by tensor product space which is a much larger space than Cartesian space. Hence the tensor product is the fundamental building operation of quantum systems that occupies a central place in the subject of many body quantum mechanics.

Stimulus	Response	Property/ Coefficient	Scalar Equation	Tensor Equation	Tensor Rank
Temperature (ΔT)	Electric Polarization (P_α)	Pyroelectricity (p or p_α)	$P = p.\Delta T$	$P_\alpha = p_\alpha \Delta T$	First p_α
Temperature (ΔT)	Strain ($\varepsilon_{\alpha\beta}$)	Thermal expansion (a or $a_{\alpha\beta}$)	$\varepsilon = a.\Delta T$	$\varepsilon_{\alpha\beta} = a_{\alpha\beta}\Delta T$	Second $a_{\alpha\beta}$
Stress ($\sigma_{\beta\gamma}$)	Elec. Polarization (P_α)	Elec.Polarization (b or $b_{\alpha\beta\gamma}$)	$P = b.\sigma$	$P_\alpha = b_{\alpha\beta\gamma}\sigma_{\beta\gamma}$	Third $b_{\alpha\beta\gamma}$
Strain ($\varepsilon_{\gamma\delta}$)	Stress ($\sigma_{\alpha\beta}$)	Elasticity (c or $c_{\alpha\beta\gamma\delta}$)	$\sigma = c.\varepsilon$	$\sigma_{\alpha\beta} = c_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}$	Fourth $c_{\alpha\beta\gamma\delta}$

TABLE 3. Few material property tensor coefficients with their stimulus and response relation.

4.4 Entanglement

A major concept setting quantum reality apart from the classical one is the notion of entanglement. The fact that physical systems can be correlated in ways that exceed shared randomness of classically correlated systems is an important asset in the field of quantum computation. Quantum computation is a fast developing field, at the interface of Quantum mechanics, Computer science and Information theory, that tries to harness this weird aspect of quantum reality for technological purposes. Quantum computers are not limited to two states like present day computers which work by manipulating bits that exist in one of two states, $|0\rangle$ or $|1\rangle$. Quantum computers encode information as quantum bits or qubits that can exist in superposition of both $|0\rangle$ and $|1\rangle$ states at the same time. The tensor product succinctly captures this same distinct quantum behavior as has been discussed in the above section.

But if the states of two distinct quantum systems cannot be factorised as a tensor product of a wave function from one space with that from the other then the state is said to be "entangled". Thus the tensor product formalism comes into picture whenever entanglement is in consideration. The essence of quantum entanglement lies in the fact that there exist states in the tensor product space of physically separate systems that cannot be decomposed as tensor product of states from separate systems. In other words there exist states of the combined system that cannot be expressed in terms of definite states of the individual systems.

For example, suppose both V_A and V_B are two dimensional Hilbert spaces describing spin-1/2 degrees of freedom of two quantum particles. Each space can be spanned by an orthonormal basis of couple of states $|v_{Ai}\rangle \equiv |v_{Bi}\rangle = \{|0\rangle, |1\rangle\}$, representing spin-up and spin-down states. Then the tensor product space $V = V_A \otimes V_B$ is a four dimensional space spanned by pairwise basis of vectors drawn from V_A and V_B bases, that is $\{|v_{Ai}\rangle \otimes |v_{Bi}\rangle\} = \{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$. That is any state of the combined system is completely specified by $|v_{Ai}\rangle \otimes |v_{Bi}\rangle \in V_A \otimes V_B$. But as remarked above the opposite is not always true, that there exist states of the combined states which cannot be expressed as the tensor product of individual systems. For example consider the following quantum state of two spin 1/2 systems, $|\Phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$. There is no choice of basis that allows this state to be expressed in terms of separate individual particle states, i.e., $|\Phi\rangle \neq \frac{1}{\sqrt{2}}(|\Phi\rangle \otimes |\Phi\rangle)$. In such a case the two particles are in entangled state.

4.5 Data Storage and Mining

A big reason as to why tensors have now become ubiquitous in pure and applied sciences is because of the fact that they provide a means to organize multi-dimensional data, as tensors of higher rank can be represented as multi-dimensional arrays or matrices. Matrices are versatile objects that can be used for storing and accessing information in a systematic way. So in context of machine and deep learning tensors can be regarded as huge multi-dimensional containers having natural representation for data storage and data mining. It should be noted that though a **tensor** is often construed as a generalized matrix but every **matrix** is not a tensor. For instance any tensor of rank 2 can be cast in a matrix of order 2×2, but the converse that every matrix of 2×2 order is a rank-2 tensor is not true. The numerical values of the components of a tensor in its matrix representation depend on the transformation rules employed. The numerical values change when a transformation is made. Though the tensor remains invariant on coordinate transformation but its component do not and hence tensor despite appearing as an static entity are dynamical objects in essence. This dynamical aspect of tensors is what that distinguishes it from a mere matrix.

The *Figure 2* shows tensors of zero, first, second and third rank as single and multi-dimensional arrays or matrices. The availability of cheap and high computational power and storage devices has enabled extensive computations on vast amounts of data. Data mining is the process of extracting valuable knowledge or information from a large set of data. If tensor product is a useful operation for building large quantum states from sub quantum systems then tensor decomposition is a highly important tool for summarization and analysis of data. Much of the literature on data mining deals with tensor decomposition methods which are outside the scope of this article.

4.6 Non-Tensors

Finally, a brief note on non-tensors will be in order. Not every physical/mathematical quantity represented in symbols adorned with indices is a tensor. A trivial example is the components of a vector which is coordinate dependent. The individual components are not tensors because under vector transformation the components of the vector vary from system to system but in a way that the vector itself remains intact. Sometimes an entity is called a *qualified tensor* because it behaves as a tensor under a certain subclass of coordinate transformations. For instance, the differential element dx^i containing spatial components is a tensor in all the three dimensional Euclidean spaces but not in the four dimensional Minkowski space. Similarly the coordinates $\{x^i\}$ are not tensors as it makes no sense to add coordinates, for instance, adding the spherical coordinates of two points. In contrast to coordinates, the differentials dx^i are tensors. Also the Jacobian, $J^i_j = \left\{\frac{dx^i}{dx^j}\right\}$, used to quantify the changes in the infinitesimal lengths, areas, volumes, etc., that occur when changing the basis of a coordinate system, is also a non-tensor. This is because the Jacobian matrix is not defined *per se*, but is only defined with reference to two chosen coordinate systems. A tensor is an abstract entity which exists even if no basis has been referred to. The other obvious examples of non-tensors are the partial derivatives and also the Christoffel symbols. The partial/ordinary derivative does not in general yield a tensor because the derivative has no meaning outside the reference frame in which it is differentiated. For instance, if the derivative of a tensor in a coordinate system is zero, then it is not necessarily zero in other coordinate systems too. But the same does not hold for covariant derivatives, as the covariant derivatives are tensors, and if the covariant derivative vanishes in one frame, then it necessarily vanishes in all frames. Complete differentiation or covariant differentiation requires taking not only the component term but also the base vectors which are also spatially dependent, except in the case of orthonormal Cartesian coordinate system. The additional term that is added to the usual partial derivatives to make it covariant are called the Christoffel symbols. Not to mention the Christoffel symbols vanish in Cartesian coordinates.

5 Conclusions

Tensors are in essence abstract mathematical objects with deep and profound implications. Soon after the formulation of general theory of relativity the subject became popular and developed further rapidly in as much that it now appears in all branches of science and engineering. The key attribute of tensors is the facility to express invariance under appropriate transformation laws in either mathematical or physical properties/laws. But besides this fundamental property of invariance four other innate potentialities possessed by tensors comes in handy to express various aspects of physical reality in science namely anisotropicity, many particle quantum states, entanglement and big data storage capacity. Two extremely useful operations on tensors are tensor product and tensor decomposition. While tensor products are immensely relevant in quantum mechanics and quantum computation, tensor decomposition methods are highly needed in machine learning and deep learning. Therefore the bottom line is tensor is an incredibly important subject with fascinatingly significant features and amazingly wide implications.

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References

[1] Jeevan Nadirji (2015).
An Introduction to Tensors and Group Theory, 2nd Edition (Birkhäuser).

[2] Sadri Hassani (1999).
Mathematical Physics – A Modern Introduction to Its Foundations, (Springer).

[3] Robert E. Newnham (2005).
Properties of Materials: Anisotropy, Symmetry, Structure (Oxford University Press).

[4] Zafar Ahsan (2005).
Tensors: Mathematics of Differential Geometry and Relativity (Prentice Hall of India).

[5] Albert Tarantola.
Tensors for Beginners,
`http://www.ipgp.fr/~tarantola/Files/Professional/Teaching/Diverse/TensorsForBeginners/Text/Tensors.pdf`

[6] Landsberg, J. M.
Tensors: Geometry and Applications, American Mathematical Society, Providence, Rhode Island.

[7] L. Cammoun et al. (2009).
Tensors in Image Processing and Computer Vision (Springer-Verlag London).

[8] Joseph C. Kolecki.
An Introduction to Tensors for Students of Physics and Engineering, National Aeronautics and Space Administration.
`https://www.grc.nasa.gov/www/k-2/Numbers/Math/documents/Tensors_TM2002211716.pdf`