A New Approach To Polynomial Algebra And Deformations

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Abstract

Ordinary differential equations are expressed using a novel operator. A new type of polynomial algebra is introduced that is seen to occur in a wide number of ordinary differential equations. We use the novel approach to classify differential equations. Deformations of the underlying algebra are also investigated.

1 Introduction

A single variable Linear Differential Equation can be cast in the form [1]

$$[F(D) + P(x, \frac{d}{dx})]y(x) = Q(x) \quad (1)$$

where

$$F(D) = \sum_n a_n D^n \quad (2)$$

where

$D = x\frac{d}{dx}$ is a novel operator that we shall be using

$Q(x)$ gives the 'source term' of the differential equation

a_n are some x-independent constants and

P is an arbitrary operator/polynomial of x (denoted as $\sum_k P_k$, k=1,2,....) comprising of powers of x, derivatives with respect to x and other operators not expressible in the form $F(D)$. In general we indicate by P_k an operator that has net index 'k' (i.e the value by which the power of the variable is raised after the operator acts on a monomial). For instance x^4or $3x^5\frac{d}{dx}$ both are P_{+4} or P_4 terms. As a concrete example if we consider Hermite's differential equation

$$[x\frac{d}{dx} - n - \frac{1}{2}\frac{d^2}{dx^2}]H_n(x) = 0 \quad (3)$$

we can define

$F(D) = x\frac{d}{dx} - n = D - n$ and
$P(x, \frac{d}{dx}) = -\frac{1}{2}\frac{d^2}{dx^2}$.

1.1 A Specific Case

Let us concentrate on a specific case of the type of polynomial algebra where the exponent of the monomial is changed at the most by unity, Hence we need to have three type of terms

P_+ in which the exponent is increased by +1

P_0 in which the exponent is left unaltered and

P_{-1} in which the exponent is decreased by +1

We denote the algebra as $\{P_+, P_-, P_0\}$.

The rationale for restricting ourselves to this kind of algebra is that most differential equations of physical systems are of at most second order.

The polynomial algebra is defined by the two relations:

$$\begin{aligned} [P_0, P_\pm] &= \pm P_\pm \\ [P_+, P_-] &= f\left(P_0\right) \end{aligned} \tag{4}$$

Here $f(P_0)$ is a polynomial in P_0 and $P_\pm$ denote terms in P that increase or decrease the power of x by unity. We can obtain a second function $g\left(P_0\right)$ such that

$$f\left(P_0\right) = g\left(P_0\right) - g\left(P_0 - 1\right) \tag{5}$$

This equation clearly suggests that $g\left(P_0\right)$ has order one higher than $f\left(P_0\right)$. As we shall see, this relation is fundamental in defining the order of the algebra. In view of differential equations and the present algebra that we just postulated, let us put forward a new algebra that we see in certain classes of differential equations.

1.2 The New Algebra: $\{P_+, P_-, P_0\}$

We now focus on the new type of algebra that we had mentioned. It occurs in some specific differential equations. As we shall see this type of algebra occurs quite naturally for some well known class of differential equations (like the Heun class).

We consider the solutions to differential equations to be in the form of monomials. By monomials we mean an algebraic expression that has only one term. Some common examples of monomials are x^4, $3x^2$, $5x^4y^9$ etc.

Let the differential equation that we are studying be of the form:

$$\left[f_1(x)\frac{d^2}{dx^2} + f_2(x)\frac{d}{dx} + f_3(x)\right] y(x) = 0 \tag{6}$$

Keeping in mind the underlying algebra of the operators we have to take the $f_i(x)$ as

$$\begin{aligned} f_1(x) &= a_0x^3 + a_1x^2 + a_2x + a_3 \\ f_2(x) &= a_4x^2 + a_5x + a_6 \\ f_3(x) &= a_7x + a_8 \end{aligned} \tag{7}$$

which needs to be farther restricted to avoid 'unwanted' terms. In order to have the specific algebra and no other term, we must impose $a_3 = 0$. This gives three types of terms

$$\begin{aligned} P_+ &= a_0x^3\tfrac{d^2}{dx^2} + a_4x^2\tfrac{d}{dx} + a_7x \\ P_0 = F(D) &= a_1x^2\tfrac{d^2}{dx^2} + a_5x\tfrac{d}{dx} + a_8 \\ P_- &= a_2x\tfrac{d^2}{dx^2} + a_6\tfrac{d}{dx} \end{aligned} \tag{8}$$

We rewrite 8 using

$$P_0 = x\frac{d}{dx} = D \tag{9}$$

Rewriting $P_\pm$ in terms of P_0 we obtain

$$[P_+, P_-] = \alpha_1 P_0^3 + \beta_1 P_0^2 + \gamma_1 P_0 + \delta_1 = f\,(P_0) \tag{10}$$

A Casimir operator is defined as an operator that commutes with all the generators. It is defined by

$$C = P_- P_+ + g(P_0) \tag{11}$$

Explicitly working out the Casimir operator it comes out to be

$$C = a_6 a_7 + e \tag{12}$$

We obtain

$$\begin{aligned} \alpha_1 &= -4a_0 a_2 \\ \beta_1 &= 6a_0 a_2 - 3a_2 a_4 - 3a_0 a_6 \\ \gamma_1 &= (-2a_0 a_2 + 3a_0 a_6 - 2a_4 a_6 - 2a_2 a_7 + a_2 a_4) \\ \delta_1 &= -a_6 a_7 \end{aligned} \tag{13}$$

2 Deformations of Polynomial Algebras

As we saw in the last Section, general second order differential equations yield an algebra of $\{P_+, P_0, P_-\}$ type under specific conditions. In the present section we shall delve deeper into this type of algebra.

We now restrict ourselves to differential equations of the form

$$[P_+ + P_0 + P_-]\, y(x) = 0 \tag{14}$$

Here $g(P_0) = g(D)$ is such that

$$f(D) = g(D) - g(D-1) \tag{15}$$

where

$$f(D) = [P_+, P_-] \tag{16}$$

It is hence clear that $g(D)$ must have an order one higher than that of $f(D)$.

When $P_+$and P_-do not commute we get $f(D) \neq 0$ and we say that the algebra is deformed. Depending upon the highest power of D we define the degree of deformation of the algebra.

For instance, if we obtain

$$[P_+, P_-] = f(P_0) = aP_0^2 + bP_o + c \tag{17}$$

where a, b, c are constants, we say that it is a quadratic deformation of the algebra.

Since $f(D)$ is quadratic in D, clearly $g(D)$ must be cubic in D.

Taking $g(D) = \alpha D^3 + \beta D^2 + \gamma D + \delta$ we obtain upon comparison with 17 and inserting into 11

$$C = P_- P_+ + h_1(P_0) \tag{18}$$

where

$$h_1(P_0) = \frac{a}{3} P_0^3 + \frac{a+b}{2} P_0^2 + \frac{a+3b+6c}{6} P_0 + \delta \tag{19}$$

Substituting for $P_\pm$ we obtain the Casimir, though it is not of interest to us presently.

For the next higher order, if we obtain

$$[P_+, P_-] = f(P_0) = aP_0^3 + bP_o^2 + cP_0 + d \tag{20}$$

where a, b, c, d are constants, we say that it is a cubic deformation of the algebra.

As before, explicitly working out the Casimir one obtains

$$C = P_- P_+ + h_2(P_0)$$

where

$$h_2(P_0) = \frac{a}{4}P_0^4 + \left(\frac{b}{3} + \frac{a}{2}\right) P_0^3 + \left(\frac{a}{4} + \frac{b}{2} + \frac{c}{2}\right) P_0^2 + \left(\frac{b}{6} + \frac{c}{2} + d\right) P_0 \tag{21}$$

For higher order deformations we can explicitly work out following the same technique.

3 Conclusion

We find applications of the novel approach to differential equations in different cases like the Harmonic Oscillator [1], the Sextic Oscillator [1], the Heun differential equation[2][3] to name a few. The deformations that we have explored in this article seem to be novel results and possibly can be helpful in areas where algebraic structures of differential equations are dealt with.

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