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First images of the surface of Mars taken by India's Mars Orbiter Mission

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In this Issue

EDITORIAL

It is a pleasure to publish the issue 30.3 of Physics Education. Papers concern some basic concepts in electrodynamics and quantum mechanics and touch upon optics and materials phases of current interest like graphene.

I must mention the novel article by C. K. Raju on Functional differential equations.

I wish you a happy reading!

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Maxwell's Stress Tensor and Momentum Conservation in Electromagnetic Field

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Abstract

Maxwell's Stress Tensor $\widehat{\boldsymbol{\mathcal{T}}}$ owes its origin to the notion prevailing before the advent of relatvity that 'action at a distance' is actually a mechanical interaction, like push and pull, and is transmitted by an assumed mechanical property of the *Aether* which pervaded all space, in particular vacuum. Even after withrawal of Aether this tensor has a useful role to play not only in formulating conservation of momentum in a time varying electromagnetic field, but also in simplifying several problems in electrostatics and magnetostatics, by removing the distinction between the field caused by 'external sources' and the total field surrounding a distribution of charges and currents. This tensor is to be constructed on the principle that $\mathbf{f}_s(\mathbf{r}) = \boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}(\mathbf{r})$, where $\mathbf{f}_s(\mathbf{r})$ is the force acting on unit volume of a distribution of electric charges and currents. Our derivations of the stress tensors $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}$, $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})}$ and $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})}$, corresponding to electrostatic field, magnetostatic field, and time varying electromagnetic field respectively, are based on a single vector identity and application of Maxwell's field equations. We have worked out two examples of how the force on an isolated system can be calculated by surrounding it with a sphere of some radius r and integrating the stress vector over the entire surface, namely, an isolated electric charge in the electrostatic field of another charge, and an isolated magnetic dipole in the magnetostatic field of another magnetic dipole. We have taken the stress tensor to its logical end by writing momentum conservation in a time varying electromagnetic field, and then identifying $-\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})}$ as the momentum flux density $\widehat{\boldsymbol{\Phi}}$ of the field. For the special case of a pure radiation field, $\widehat{\boldsymbol{\Phi}} = -\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})} = \boldsymbol{\Pi}\, \mathbf{c}$, where $\boldsymbol{\Pi}$ is the momentum density and $\mathbf{c}$ is the 'velocity' of light. At the beginning of this article we have given a mathematical introduction to tensors, in particular stress tensors.

1 Introduction

'Action at a distance' (AAD) was an enigma to natural philosophers, from Rene Descartes[1] (1596-1650) to James Clerk Maxwell (1831-1879). We find a an account of the evolution of physical concepts in Whittaker [1]. According the Descartes, space was a plenum, a medium called *aether*, capable of transmitting force on material bodies. "It was to be regarded as the solitary tenant of the universe, save for that infinitesimal fraction of space which is occupied by ordinary matter."

Subsequent theoretical physicists and mathematicians, Robert Hooke (1635-1703), Isaac Newton (1642-1727), Reimann (1826-1866), W.Thomson (1824-1907), Maxwell and others lent their support to this view. Implicit in their belief was the assumption that *force cannot be transmitted except by actual pressure or impact.* AAD was a taboo, as abhorrent as witchcraft: I wave my hand here and a fire is ignited there. In order to support their faith in aether they contrived every possible idea, any possible mechanical model, to make aether viable.

According to Newton "All space is pervaded by an elastic medium or aether, which is capable of propagating vibrations in the same way as air propagates the vibrations of sound. This aether pervades the pores of all material bodies, and is the cause of their cohesion; its density varies from one body to another, being greatest in the interplanetary space."

Maxwell inherited this legacy. We shall quote a few passages from his celebrated paper '*A Dynamical Theory of the Electromagnetic Field*' read to the Royal Society of London on December 8, 1964[2].

> "(1) In this way mathematical thories of statical electricity, of magnetism, of the mechanical action between between conductors carrying currents, and of the induction of currents have been formed. *In these theories the force acting between two bodies is treated with reference only to the condition of the bodies and their relative position, and without reference to the surrounding medium.*"
>
> "(2) The *mechanical* difficulties, however, which are involved in the assumption of particles acting at a distance with forces which depend on their velocities are such as to prevent me from considering this theory as an ultimate one, though it may have been, and may yet be useful to the coordination of phenmena."
>
> "(3) The theory I propose may therefore be called a theory of the *Electromagnetic Field*, because *it has to do with the space in the neighbourhood of the electric and magnetic bodies*, and it may be called a *Dynamical* Theory, because it assumes that in that space there is matter in motion, by which the observed electromagnetic phenomena are produced."
>
> "(4) The electromagnetic field is that part of space which contains and surrounds bodies in electric and magnetic

[1]The Cartesian coordinate system is associated with his name

coditions. ... It may contain any kind of matter, or *we may render it empty of all gross matter*, as in the case of Geissler's Tubes and other so called *vacua*.

There is always, however, enough matter to receive and transmit the undulations of light and heat, and it is because of the transmission of these radiations is not greatly altered when transparen bodies of measurable densities are substituted for the so-called vacuum, that we are obliged to admit that the *undulations are those of aetherial substance*, and not of the gross matter, the presence of which merely modifies in some way the motion of the aether.

We have therefore some reason to believe, from the phenomena of light and heat, that *there is an aetherial medium filling space and permeating bodies, capable of being set in motion and of transmitting that motion from one part to another, and communicating that motion to gross matter so as to heat it and affect it in various ways.* "

One aspect of the mechanical model Maxwell built up to present a complete picture of the electromagnetic field was the proposition that space, i.e., aether, can sustain stress, and a force is transmitted from one body (electrified or magnetized) to another by means of stress, in the same way a force is transmitted from one end of a cable to the other by means of tensile stress, and from one part of a beam to another by means of shear stress.

In his two-volume book '*A treatise on Electricity and Magnetism*' Maxwell presents a complete formulation of the Stress in the field (read aether) by constructing the *Stress Tensor* for the *Static Electric Field*[3] and for the *Static Magnetic Field*[4], in terms of the field potentials. The first one is presented in Vol 1 of his book and second one in Vol 2. His derivation of the first tensor (for the Electrostatic field) involves manipulation of Laplace's and Poisson's equations. His derivation of the second tensor (for the Magnetostatic field) involves magnetic poles which are now out of fashion in current physics text books, and may not be of much interest to us.

We have derived the stress tensors for Electrostatic field, Magnetostatic field and time varying Electromagnetic field in terms of the electric field $\mathbf{E}$, magnetic field $\mathbf{B}$ in a unified manner exploiting the useful identity given in Eq. (76).

Einstein's formulation of the Special Theory of Relativity saw the demise of the Luminiferous (i.e., light carrying) Aether. Light travels in empty space, eletric and magnetic forces also propagate from one body to another (with the speed of light) in empty space. Is there then any place for Maxwell's Stress Tensor? Is it only for historical reason that we are writing this long article? We shall attempt to provide the answer in four steps.

First, it is indeed an amazing thing that the force acting on an isolated body A (which may consist of electric charges and currents), due to the presence of charges and currents elsewhere, can be computed *exactly* by draw-

ing a boundary surface $\mathcal{S}$ of our convenience surrounding A, as in Fig.1(a), finding the "*stress*" all over this surface, and by integrating this stress. In other words, there *is* stress even in vacuum. The purpose of this article is to articulate how this stress is to be found out. Also it should be noted with interest that even empty space is not a true vacuum. When loaded with the electric and magnetic fields space comes under stress. Empty space is always buzzing with emission and absorption of virtual particles, with the virtual photons mediating the interaction among electrified and magnetized objects. Aren't these virtual photons the new *Avatar* of the Aaether?

Secondly, calculating the force on an isolated object A requires *exact* knowledge of the $\mathbf{E}$ or $\mathbf{B}$ field in which A is immersed. In recognizing these fields one has to be very careful that these $\mathbf{E}, \mathbf{B}$ fields do not contain any trace of the fields contributed by A itself. This is sometimes a challenging task. Consider for example the force acting on the surface of a conductor carrying a suface charge density σ, as in Fig.1(b). The electric field just outside the surface is $\mathbf{E} = (\sigma/\epsilon_0)\mathbf{n}$ where $\mathbf{n}$ is a unit normal to the surface. One may be tempted to conclude that the force per unit area of the surface is $\mathbf{F}' = \sigma\mathbf{E} = (\sigma^2/\epsilon_0)\mathbf{n}$, forgetting the fact that an infinitesimal area *da* on the surface contributes the same $\mathbf{E}$ field perpendicular to the surface as the rest of the surface, so that the true force is

$$\mathbf{F} = \frac{1}{2}\mathbf{F}' = (\sigma^2/2\epsilon_0)\mathbf{n} = (\epsilon_0 E^2/2)\mathbf{n}. \qquad (1)$$

The stress tensor approach, which uses the total field $\mathbf{E}_{\text{total}}$, making no distinction between the test object and the source object, will give the right result without creating any confusion, as we shall show following Eq. (84).

Thirdly, it is always advisable to arrive at the same answer through several alternative routes, if available, just to make sure that we have not made any mistakes. The stress tensor provides that valuable alternative route.

And fourthly, *Maxwell's Stress Tensor*, which we shall denote by the symbol $\widehat{\boldsymbol{\mathcal{T}}}$, is needed for understanding conservation and flow of *momentum* in the electromagnetic field, which we shall present in Section 6. When one goes deeper into the theory of relativity the same tensor appears as the most important component of the Energy-Momentum tensor required not only for presenting a 4-dimensiinal and unified view of the conservation of energy and momentum, but also for building up the source term in formulating Einstein's field equation for the gravitational field, in his General Theory of Relativity.

Maxwell's stress tensor has been discussed in all standard books on Electrodynamics[5, 6, 7, 8, 9]. However it has received a more detailed treatment in the books by Panofsky and Phillips and Griffiths. Griffiths has worked out a very interesting problem to bring out the meaning of this tensor. In this article (See Secs. 4.3 and 5.2) we have contributed two worked out prblems to illustrate the same concept.

We shall begin this pedagogical article by giving the reader a mathematial introduction to tensor, and then specializing the same to

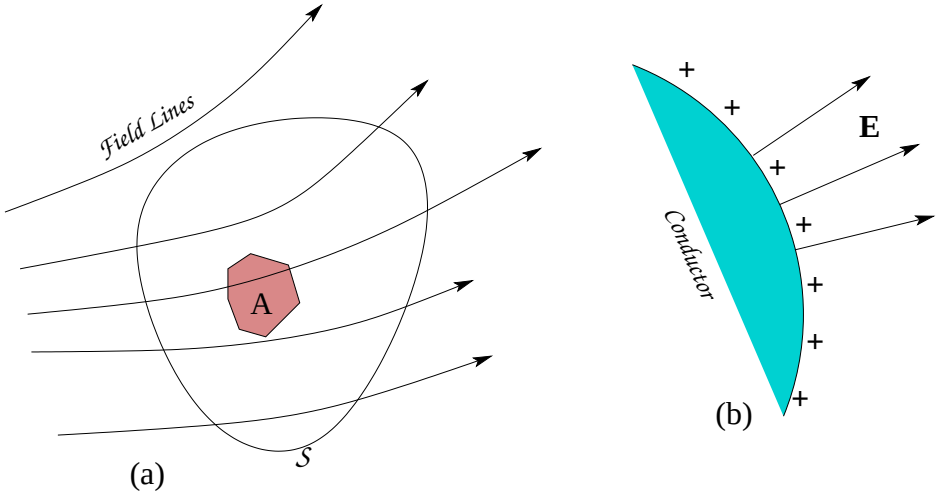

Figure 1: *Electrified object in* **E** *field.*

stress tensor, in particular Maxwell's stress tensor.

2 Introduction to Tensor

2.1 Linear Operator in a Vector Space

What we shall call a *tensor* in this book is actually "a tensor of rank 2". A bold capital letter with a "widehat" on top, e.g., $\widehat{\mathbf{T}}$, will symbolize such a tensor. In fact a scalar, e.g., potential energy V is called a tensor of rank 0, a vector, e.g., momentum $\mathbf{p}$ a tensor of rank 1.

Maxwell's stress tensor $\widehat{\boldsymbol{\mathcal{T}}}$, which is a tensor of rank 2 is needed for understanding conservation and flow of *momentum* in the electromagnetic field. in this esction we shall prepare the ground work for constructing this tensor.

We shall begin by explaining what we mean by *linear operator* in a vector space.

By the 3-dimensional linear vector space $\mathcal{V}$ we mean the set of all vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}, \cdots$ we can think of and all such vectors we can construct by combining them linearly, e.g., $\eta\mathbf{A} + \lambda\mathbf{B}$ where η, λ are real numbers.

Let us think of two vectors $\mathbf{C}$ and $\mathbf{D}$ having Cartesian components (C_x, C_y, C_z) and (D_x, D_y, D_z) and related to each other in such a way that the values of the former determine the values of the latter. This means that $\mathbf{C}$ is an independent vector and $\mathbf{D}$ is a dependent one. In other words $\mathbf{D}$ is a function of $\mathbf{C}$. Let us further assume that $\mathbf{D}$ is proportional to $\mathbf{C}$. That is, if for example we double $\mathbf{C}$, then $\mathbf{D}$ is doubled. These two vectors, however, may or may not be in the same direction. In that case we say that a *linear operator* $\widehat{\mathcal{O}}$ transforms $\mathbf{C}$ into $\mathbf{D}$. We may like to write

this transformation symbolically as

$$\widehat{\mathcal{O}}(\mathbf{C}) = \mathbf{D}. \tag{2}$$

The property of linearity means that

$$\begin{aligned} &\text{If} \quad \widehat{\mathcal{O}}(\mathbf{C}) = \mathbf{D}, \text{ and } \quad \widehat{\mathcal{O}}(\mathbf{E}) = \mathbf{F}, \\ &\text{then} \quad \widehat{\mathcal{O}}(a\mathbf{C} + b\mathbf{E}) = a\mathbf{D} + b\mathbf{F}, \end{aligned} \tag{3}$$

where a, b are two arbitrary scalar constants.

In Fig.2 we have shown two simple examples of how the operation $\widehat{\mathcal{O}}$ can take place. In Fig.(a) we have shown a particle of constant mass m in abrbtrary motion along some trajectory Γ. At some instant of time t it has velocity $\mathbf{v}$. Therefore its momentum at the same instant is $\mathbf{p} = m\mathbf{v}$. We can therefore think of the operator $\widehat{\mathcal{O}}$ transforming velocity $\mathbf{v}$ into momentum $\mathbf{p}$ by scaling the length of the former by the factor m without changing its direction.

In Fig.(b) we have shown a Rigid Body rotating about some axis pointing in the direction of the unit vector $\mathbf{n}$ with angular speed ω, so that its angular velocity is $\boldsymbol{\omega} = \omega\mathbf{n}$. Its angular momentum is $\mathbf{L}$, which (in general) does not coincide with the direction of $\boldsymbol{\omega}$. In this case the operator $\widehat{\mathcal{O}}$ transforms the angular velocity $\boldsymbol{\omega}$ into angular momentum $\mathbf{L}$ by changing the length as well as the direction. The linear operator $\widehat{\mathcal{O}}$ in this case is the inertia tensor $\widehat{\boldsymbol{\mathcal{I}}}$ about which we shall give some more insight in Sec.2.4.

For our immediate purpose we shall look upon a tensor $\widehat{\mathbf{T}}$ as a linear operator. The linear operation mentioned above suggests that $\widehat{\mathbf{T}}$ can be represented by a matrix, and the "tensor operation" can be represented as a matrix multiplication. This will become evident in the next section.

2.2 Tensor as a Dyadic

Two arbitrary vectors $\mathbf{A}, \mathbf{B}$ can be combined in three types of "multiplication operation", the first two of which the reader is familiar with, namely, (1) the dot product $\mathbf{A}\cdot\mathbf{B}$ which is a scalar; (2) the cross product $\mathbf{A}\times\mathbf{B}$ which is a vector. Now comes (3) the third type, namely the *dyadic product* $\mathbf{AB}$, which is a simple juxtaposition of the vectors, without any dot or cross in between, which we shall call a *dyad.*

We define the *dyad* $\mathbf{AB}$ to be a *linear operator* which converts any vector $\mathbf{C}$ to another vector $\mathbf{D}$ and this conversion can be done in either of the following two ways:

$$\begin{aligned} &(a) \text{ operating } \textit{on the right}: \\ &\mathbf{AB}\cdot\mathbf{C} \stackrel{\text{def}}{=} \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) = \eta\mathbf{A} \\ &\qquad \text{where } \eta = \mathbf{B}\cdot\mathbf{C} = \text{scalar}. \\ &(b) \text{ operating } \textit{on the left}: \\ &\mathbf{C}\cdot\mathbf{AB} \stackrel{\text{def}}{=} (\mathbf{C}\cdot\mathbf{A})\mathbf{B} = \lambda\mathbf{B} \\ &\qquad \text{where } \lambda = \mathbf{C}\cdot\mathbf{A} = \text{scalar}. \end{aligned} \tag{4}$$

The linearity property follows from the operation defined in (4). Also note that in general, $\mathbf{AB} \neq \mathbf{BA}$.

We shall write the sum of two dyads $\mathbf{AB}$ and $\mathbf{EF}$ as $\mathbf{AB} + \mathbf{EF}$ and define it by the distributive property:

$$\begin{aligned} (\mathbf{AB}+\mathbf{EF})\cdot\mathbf{C} &\stackrel{\text{def}}{=} \mathbf{AB}\cdot\mathbf{C} + \mathbf{EF}\cdot\mathbf{C} \\ &= \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) + \mathbf{E}(\mathbf{F}\cdot\mathbf{C}). \\ \mathbf{C}\cdot(\mathbf{AB}+\mathbf{EF}) &\stackrel{\text{def}}{=} \mathbf{C}\cdot\mathbf{AB} + \mathbf{C}\cdot\mathbf{EF} \\ &= (\mathbf{C}\cdot\mathbf{A})\mathbf{B} + (\mathbf{C}\cdot\mathbf{E})\mathbf{F}. \end{aligned} \tag{5}$$

It should be a simple exercise to show from Eqs. (4) that the dyadic product is distribu-

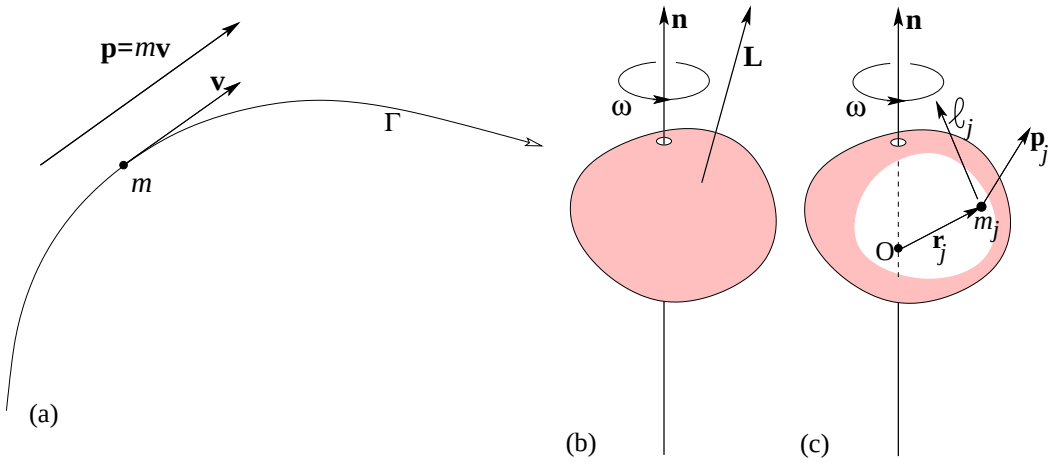

Figure 2: *two examples of how a linear operator $\widehat{\mathcal{O}}$ transforms a vector into another vector: (a) $\widehat{\mathcal{O}}$ acting on* $\mathbf{v}$ *yields* $\mathbf{p}$*; (b,c) $\widehat{\mathcal{O}}$ acting on* $\boldsymbol{\omega}$ *yields* $\mathbf{L}$.

tive, i.e., if $\mathbf{E}, \mathbf{F}, \mathbf{C}$ are three arbitrary vectors, then

$$\begin{aligned} (\mathbf{E}+\mathbf{F})\mathbf{C} &= \mathbf{EC}+\mathbf{FC}. \\ \mathbf{C}(\mathbf{E}+\mathbf{F}) &= \mathbf{CE}+\mathbf{CF}. \end{aligned} \tag{6}$$

As a corollary,

$$(\mathbf{A}+\mathbf{B})(\mathbf{E}+\mathbf{F}) = \mathbf{AE}+\mathbf{AF}+\mathbf{BE}+\mathbf{BF}. \tag{7}$$

A sum of dyads can be called a *dyadic*. We shall prefer to use the term "dyadic" as a general name for sums of dyads as well as individual dyads.

We shall frequently use the symbols $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ to represent unit vectors in the directions of the X, Y, Z axes, for which we had used $\mathbf{i}, \mathbf{j}, \mathbf{k}$ earlier in this chapter. As we progress we shall use another set of symbols $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to mean the same unit vectors. This transition $(\mathbf{i}, \mathbf{j}, \mathbf{k}) \rightarrow (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) \rightarrow (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, side by side with $(x, y, z) \rightarrow (x_1, x_2, x_3)$ will restore symmetry and help us use Einstein's summation convention (following Eq.9).

Let us now consider the set of 12 dyads: $\{\mathbf{e}_x\mathbf{e}_x, \mathbf{e}_x\mathbf{e}_y, \mathbf{e}_x\mathbf{e}_z, \cdots, \mathbf{e}_z\mathbf{e}_z\}$. Using them we can construct the following dyadic

$$\begin{aligned} \widehat{\mathbf{T}} &= T_{xx}\mathbf{e}_x\mathbf{e}_x + T_{yx}\mathbf{e}_y\mathbf{e}_x + \cdots \\ &\quad + T_{yz}\mathbf{e}_y\mathbf{e}_z + T_{zz}\mathbf{e}_z\mathbf{e}_z \\ &= \textstyle\sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}\mathbf{e}_i\mathbf{e}_j \equiv T_{ij}\mathbf{e}_i\mathbf{e}_j, \end{aligned} \tag{8}$$

where the subscripts (1,2,3) represent (x, y, z) respectively. That is

$$\begin{aligned} &\mathbf{e}_1 \equiv \mathbf{e}_x; \quad \mathbf{e}_2 \equiv \mathbf{e}_y; \quad \mathbf{e}_3 \equiv \mathbf{e}_z; \\ &\text{and,} \quad T_{11} \equiv T_{xx};\ T_{12} \equiv T_{xy}; \cdots; \\ &\qquad T_{32} \equiv T_{zy};\ T_{33} = T_{zz} \end{aligned} \tag{9}$$

are arbitrary real numbers.

In the second line of Eq. (8) we have introduced *Einstein's summation convention*: *sum over repeated index*, without explicitly inserting the sum symbol $\sum$. The subscript "$_i$" ap-

pears twice, implying a sum over i. The subscript "$_j$" appears twice, implying one more sum, this time over j.

The mathematical object $\widehat{\mathbf{T}}$ appearing in Eq. (8) is what we shall call a *tensor* for all purposes in this book. The set of dyads $\{\mathbf{e}_x\mathbf{e}_x,\, \mathbf{e}_x\mathbf{e}_y,\, \mathbf{e}_x\mathbf{e}_z,\, \cdots,\, \mathbf{e}_z\mathbf{e}_z\}$ can be looked upon as a complete set of base dyads forming a *basis* $\widehat{\mathcal{B}}$ in the tensor space $\mathfrak{T}$ of $\widehat{\mathbf{T}}$. This is analogous to the way that the vectors $\{\mathbf{e}_x,\, \mathbf{e}_y,\, \mathbf{e}_z\}$ form a *basis* $\mathcal{B}$ in the vector space $\mathfrak{V}$ of $\mathbf{V}$. Any arbitrary vector $\mathbf{V}$ can be written as a linear superposition of the base vectors as

$$\begin{aligned} &\mathbf{V} = V_x\mathbf{e}_x + V_y\mathbf{e}_y + V_z\mathbf{e}_z, \quad (a)\\ \text{where}\quad &V_x = \mathbf{V}\cdot\mathbf{e}_x,\, V_y = \mathbf{V}\cdot\mathbf{e}_y,\\ &V_z = \mathbf{V}\cdot\mathbf{e}_z, \qquad\qquad (b) \end{aligned} \tag{10}$$

are the Cartesian (scalar) components of $\mathbf{V}$ in the basis $\mathcal{B}$. In the same way any arbitary tensor $\widehat{\mathbf{T}}$ can be written as a linear superposition of the base dyads, as in Eq. (8), where the nine quantities $\{T_{xx}, T_{xy}, \cdots T_{zy}, T_{zz}\}$ are to be interpreted as the Cartesian (scalar) *components* of $\widehat{\mathbf{T}}$ with respect to this basis $\widehat{\mathcal{B}}$.

From the definition of dyad given in (4), and the orthogonality of the base vectors $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, i.e.,

$$\mathbf{e}_j\cdot\mathbf{e}_k = \delta_{jk}; \quad j,k = 1,2,3 = x,y,z, \tag{11}$$

it should be apparent that the base dyads operating on any arbitrary vector $\mathbf{V}$ will yield the following vectors:

$$\begin{aligned} &\mathbf{e}_x\mathbf{e}_x\cdot\mathbf{V} = \mathbf{e}_xV_x; \quad \mathbf{e}_x\mathbf{e}_y\cdot\mathbf{V} = \mathbf{e}_xV_y; \ \cdots\\ &\mathbf{e}_z\mathbf{e}_y\cdot\mathbf{V} = \mathbf{e}_zV_y; \quad \mathbf{e}_z\mathbf{e}_z\cdot\mathbf{V} = \mathbf{e}_zV_z.\\ &\mathbf{V}\cdot\mathbf{e}_x\mathbf{e}_x = V_x\mathbf{e}_x; \quad \mathbf{V}\cdot\mathbf{e}_x\mathbf{e}_y = V_x\mathbf{e}_y; \ \cdots\\ &\mathbf{V}\cdot\mathbf{e}_z\mathbf{e}_y = V_z\mathbf{e}_y; \quad \mathbf{V}\cdot\mathbf{e}_z\mathbf{e}_z = V_z\mathbf{e}_z. \end{aligned} \tag{12}$$

Hence, if $\mathbf{A} = A_x\mathbf{e}_x + A_y\mathbf{e}_x + A_z\mathbf{e}_z$ and $\mathbf{B} = B_x\mathbf{e}_x + B_y\mathbf{e}_x + B_z\mathbf{e}_z$ are two arbitrary vectors, then,

$$\begin{aligned} &\mathbf{A}\cdot\widehat{\mathbf{T}}\cdot\mathbf{B} \stackrel{\text{def}}{=} \mathbf{A}\cdot(\widehat{\mathbf{T}}\cdot\mathbf{B})\\ &= A_iT_{ij}B_j = (\mathbf{A}\cdot\widehat{\mathbf{T}})\cdot\mathbf{B}. \quad (a)\\ &\text{As a special case}\\ &\mathbf{e}_i\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_j = T_{ij}. \qquad (b) \end{aligned} \tag{13}$$

If the nine components $\{T_{ij}\}$ of a tensor $\widehat{\mathbf{T}}$ are given, the tensor can be constructed using Eq. (8). Conversely, if a tensor $\widehat{\mathbf{T}}$ is given in the form of a mathematical relation, its nine components T_{ij} can be retrieved by means of Eq. (13b).

Using the distributive property given in (7) it is seen that the dyadic product of $\mathbf{A}$ and $\mathbf{B}$ has the following dyadic representation:

$$\begin{aligned} \mathbf{AB} &= A_xB_x\mathbf{e}_x\mathbf{e}_x + A_xB_y\mathbf{e}_x\mathbf{e}_y + \cdots\\ &\quad + A_zB_y\mathbf{e}_z\mathbf{e}_y + A_zB_z\mathbf{e}_z\mathbf{e}_z\\ &= A_iB_j\mathbf{e}_i\mathbf{e}_j. \end{aligned} \tag{14}$$

Hence, if we write

$$\widehat{\mathbf{T}} = \mathbf{AB}, \quad \text{then,} \quad T_{ij} = A_iB_j. \tag{15}$$

Using Eq. ((a)4), the operation of the tensor $\widehat{\mathbf{T}}$ on the vector $\mathbf{C} = C_k\mathbf{e}_k$ placed on the *right* works out as follows.

$$\begin{aligned} \widehat{\mathbf{T}}\cdot\mathbf{C} &= (T_{ij}\mathbf{e}_i\mathbf{e}_j)\cdot(C_k\mathbf{e}_k)\\ &= T_{ij}\,C_k\,\mathbf{e}_i(\mathbf{e}_j\cdot\mathbf{e}_k)\\ &= \mathbf{e}_i\,(T_{ij}\,C_j). \end{aligned} \tag{16}$$

We have used the orthogonality relation (11) to get to the last line.

In a similar way, using Eq. ((b)4), the operation of the tensor $\widehat{\mathbf{T}}$ on the vector $\mathbf{C} = C_k\mathbf{e}_k$ placed on the *left* works out as follows.

$$\begin{aligned}\mathbf{C}\cdot\widehat{\mathbf{T}} &= (C_k\mathbf{e}_k)\cdot(T_{ij}\mathbf{e}_i\mathbf{e}_j)\\ &= C_k\,T_{ij}\,(\mathbf{e}_k\cdot\mathbf{e}_i)\mathbf{e}_j\\ &= (C_k\,T_{kj})\,\mathbf{e}_j.\end{aligned}\qquad(17)$$

The above two equations suggest that if we write $\mathbf{D} = \widehat{\mathbf{T}}\cdot\mathbf{C}$ and $\mathbf{F} = \mathbf{C}\cdot\widehat{\mathbf{T}}$, then the Cartesian components (D_1, D_2, D_3) of $\mathbf{D}$ and (F_1, F_2, F_3) of $\mathbf{F}$ can be obtained from matrix multiplications:

$$\begin{aligned}&\begin{pmatrix} D_1\\ D_2\\ D_3\end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{pmatrix}\begin{pmatrix} C_1\\ C_2\\ C_3\end{pmatrix}.\\ &\begin{pmatrix} F_1 & F_2 & F_3\end{pmatrix}\\ &= \begin{pmatrix} C_1 & C_2 & C_3\end{pmatrix}\begin{pmatrix} T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{pmatrix}.\end{aligned}\qquad(18)$$

In the above equations, starting from Eq. (4), we have used a dot $(\cdot)$ to separate the tensor from the vector on which it is operating. We shall frequently refer to a tensor operation as a *dot product* between the tensor and the vector. Eqs. (18) show that a *dot product actually involves a matrix multiplication.* A tensor is to be represented as a *square matrix*, and a vector either as a *column matrix* or a *row matrix*, depending on whether the tensor operation is on the right or on the left[2].

$$\begin{aligned}\widehat{\mathbf{T}} &= \begin{pmatrix} T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{pmatrix} = [T];\\ \mathbf{C} &= \begin{pmatrix} C_1\\ C_2\\ C_3\end{pmatrix} = \{C\};\\ \mathbf{F} &= \begin{pmatrix} F_1 & F_2 & F_3\end{pmatrix} = (F).\end{aligned}\qquad(19)$$

In the above we have adopted the convention of indicating a 3×3 square matrix by [ ], a 3×1 column matrix by { }, and a 1×3 row matrix by ( ). Hence, Eqs.(18) can be written as

$$\{D\} = [T]\{C\};\quad (F) = (C)[T].\qquad(20)$$

It follows from Eq. (14) that the matrix representation of the dyadic $\mathbf{AB}$ is

$$\mathbf{AB} = \begin{pmatrix} A_1B_1 & A_1B_2 & A_1B_3\\ A_2B_1 & A_2B_2 & A_2B_3\\ A_3B_1 & A_3B_2 & A_3B_3\end{pmatrix}.\qquad(21)$$

We shall define the dot product of two tensors $\widehat{\mathbf{S}}$ and $\widehat{\mathbf{T}}$ as the tensor $\widehat{\mathbf{R}} = \widehat{\mathbf{S}}\cdot\widehat{\mathbf{T}}$ by its operation on an arbitrary vector $\mathbf{C}$ *on the right* in the following way.

$$(\widehat{\mathbf{S}}\cdot\widehat{\mathbf{T}})\cdot\mathbf{C} \stackrel{\text{def}}{=} \widehat{\mathbf{S}}\cdot(\widehat{\mathbf{T}}\cdot\mathbf{C}).\qquad(22)$$

[2] In Quantum Mechanics (QM) a clear distinction is made between a vector $\mathbf{A}$ on left and a vector $\mathbf{B}$ on right, as in the scalar product $\mathbf{A}\cdot\mathbf{B}$. The former is called a *bra* vector and the latter a *ket* vector, and together, in the scalar product, they constitute a *bra-ket*: $\mathbf{A}\rightarrow<A|$; $\mathbf{B}\rightarrow|B>$; $\mathbf{A}\cdot\mathbf{B}\rightarrow<A|B>$. However, these vectors are in genereral infinite dimensional, their components are complex numbers, and the components of the bra vector $<A|$ are complex conjugates of the respective components of the ket vector $|A>$.

From this it follows that the matrix representing $\widehat{\mathbf{R}}$ is given by the product of the matrices representing $\widehat{\mathbf{S}}$ and $\widehat{\mathbf{T}}$. That is,

$$[R] = [S]\,[T], \text{ implying: } R_{ij} = S_{ik}T_{kj}. \quad (23)$$

It is then obvious that, in general, $\widehat{\mathbf{S}} \cdot \widehat{\mathbf{T}} \neq \widehat{\mathbf{T}} \cdot \widehat{\mathbf{S}}$.

Using the matrix representation as given in Eq. (23), and the tensor operation on the left as found out in (17), we can now see how the product tensor $\widehat{\mathbf{R}} = \widehat{\mathbf{S}} \cdot \widehat{\mathbf{T}}$ will act *on the left.*

$$\begin{aligned} \mathbf{C} \cdot \widehat{\mathbf{R}} &= (C_k R_{kj})\mathbf{e}_j = (C_k S_{km} T_{mj})\mathbf{e}_j \\ &= (C_k S_{km})(T_{mj}\mathbf{e}_j). \quad (24) \\ \text{Or,} \quad \mathbf{C} \cdot (\widehat{\mathbf{S}} \cdot \widehat{\mathbf{T}}) &= (\mathbf{C} \cdot \widehat{\mathbf{S}}) \cdot \widehat{\mathbf{T}}. \end{aligned}$$

We can extend the definition of matrix product to any number of tensors, by writing the matrix representation of the product tensor as the product of the representative matrices of the component tensors. For example

$$\text{If } \widehat{\mathbf{R}} = \widehat{\mathbf{A}} \cdot \widehat{\mathbf{B}} \cdot \widehat{\mathbf{C}}, \text{ then } [R] = [A]\,[B]\,[C]. \quad (25)$$

At this point we shall add a word of caution. A tensor is not the same as a square matrix, just as a vector is not the same as a column matrix or a row matrix. The row matrix shown in Eq. (19), for example, gives the components of the vector $\mathbf{F}$ in a given coordinate system XYZ. As the coordinates are changed from (x, y, z) to (x', y', z'), the components will transform from (F_1, F_2, F_3) to (F_1', F_2', F_3'). However, the vector $\mathbf{F}$ itself is a "geometrical object" (a straight line of measured length pointing in an assigned direction) which remains invariant under all coordinate transformations. In the same way the tensor $\widehat{\mathbf{T}}$ is a geometrical object, which remains invariant under all coordinate transformations, even though its components will change from the square matrix $[T_{ij}]$ to another square matrix $[T_{ij}']$ under the same coordinate transformation.

Yes, the components of all tensors will transform, except the components of the identity tensor which we shall introduce in the next section. They will remain the same, the same as in (27), following any coordinte transformations.

2.3 Identity Tensor, Completeness Relation, Components of a Tensor in the Spherical coordinate system

In matrix multiplication one needs the *identity matrix* $\widehat{1}$ which in the present context, is the matrix representation of the *identity tensor*, also known by the alternative name *idemfactor*. It will be recognized by the symbol $\widehat{\mathbf{1}}$. Its sole property is that when it operates on any vector $\mathbf{V}$, either on the right, or on the left, it gives back the same vector.

$$\widehat{\mathbf{1}} \cdot \mathbf{V} \stackrel{\text{def}}{=} \mathbf{V}; \qquad \mathbf{V} \cdot \widehat{\mathbf{1}} \stackrel{\text{def}}{=} \mathbf{V}. \quad (26)$$

Such a tensor must have $\widehat{1}$ for its matrix representation. The dyadic representation (shown below) follows from the above prop-

erty and the orthogonality relation (11).

$$\begin{aligned}\widehat{\mathbf{1}} &= \widehat{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (a)\\ \widehat{\mathbf{1}} &= \mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z = \mathbf{e}_i\mathbf{e}_i. \quad (b)\end{aligned} \tag{27}$$

Eq.(a) gives the Matrix representation, and Eq.(b) the Dyadic representation.

It will be advantageous to write the tensor $\widehat{\mathbf{T}}$ in a curvilinear coordinate systen, in particular, spherical coordinate system. The reader must be familar with the following transformation equations for the coordinates and the base vectors.

$$\begin{aligned}&(x, y, z) = (r\ \sin\theta\ \cos\phi, r\ \sin\theta\ \sin\phi, r\ \cos\theta).\\ &\mathbf{e}_r = \sin\theta\,(\cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y) + \cos\theta\,\mathbf{e}_z;\\ &\mathbf{e}_\theta = \cos\theta\,(\cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y) - \sin\theta\,\mathbf{e}_z;\\ &\mathbf{e}_\phi = -\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y.\end{aligned} \tag{28}$$

Using these equations (and remembering that $\mathbf{e}_r\mathbf{e}_\theta \neq \mathbf{e}_\theta\mathbf{e}_r$, for example) it should be a simple exercise to show that

$$\mathbf{e}_r\mathbf{e}_r + \mathbf{e}_\theta\mathbf{e}_\theta + \mathbf{e}_\phi\mathbf{e}_\phi = \mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z = \widehat{\mathbf{1}}. \tag{29}$$

If we have three unit vectors $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ which are mutually orthogonal at every point in space and such that

$$\boldsymbol{aa} + \boldsymbol{bb} + \boldsymbol{cc} = \widehat{\mathbf{1}}, \tag{30}$$

then we say that these three vectors form a *complete orthogonal set*, and hence a basis, so that any arbitrary vector $\mathbf{V}$ can be represented as a linear superposition of these three vectors [3]. This should be clear from the following.

$$\begin{aligned}&\mathbf{V} = \mathbf{V}\cdot\widehat{\mathbf{1}} = \mathbf{V}\cdot(\boldsymbol{aa} + \boldsymbol{bb} + \boldsymbol{cc})\\ &= V_a\boldsymbol{a} + V_b\boldsymbol{b} + V_c\boldsymbol{c},\\ &\text{where}\quad V_a = \mathbf{V}\cdot\boldsymbol{a},\ V_b = \mathbf{V}\cdot\boldsymbol{b},\ V_c = \mathbf{V}\cdot\boldsymbol{c},\end{aligned} \tag{31}$$

are the components of $\mathbf{V}$ in the directions of $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ respectively. Using the completeness property it can be advantageous to write a tensor in the following style.

$$\begin{aligned}&\widehat{\mathbf{T}} = \widehat{\mathbf{1}}\cdot\widehat{\mathbf{T}}\cdot\widehat{\mathbf{1}}\\ &= (\boldsymbol{aa} + \boldsymbol{bb} + \boldsymbol{cc})\cdot\widehat{\mathbf{T}}\cdot(\boldsymbol{aa} + \boldsymbol{bb} + \boldsymbol{cc})\\ &= T_{aa}\boldsymbol{aa} + T_{ab}\boldsymbol{ab} + T_{ac}\boldsymbol{ac}\cdots\\ &+T_{cb}\boldsymbol{cb} + T_{cc}\boldsymbol{cc}, \quad \text{where}\\ &T_{aa} = \boldsymbol{a}\cdot\widehat{\mathbf{T}}\cdot\boldsymbol{a},\ T_{ab} = \boldsymbol{a}\cdot\widehat{\mathbf{T}}\cdot\boldsymbol{b}, \cdots,\\ &T_{cb} = \boldsymbol{c}\cdot\widehat{\mathbf{T}}\cdot\boldsymbol{b},\ T_{cc} = \boldsymbol{c}\cdot\widehat{\mathbf{T}}\cdot\boldsymbol{c}\end{aligned} \tag{32}$$

are the components of $\widehat{\mathbf{T}}$ with respect to the basis $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$.

We shall illustrate the operation shown in Eq. (32) by writing the tensor $\widehat{\mathbf{T}}$ in Cartesian and spherical coordinate systems.

$$\begin{aligned}\widehat{\mathbf{T}} &= (\mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z)\cdot\widehat{\mathbf{T}}\cdot(\mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z) &(a)\\ &= T_{xx}\mathbf{e}_x\mathbf{e}_x + T_{xy}\mathbf{e}_x\mathbf{e}_y + T_{xz}\mathbf{e}_x\mathbf{e}_z + \cdots + T_{zx}\mathbf{e}_z\mathbf{e}_y + T_{zz}\mathbf{e}_z\mathbf{e}_z, \quad \text{where} &(b)\\ T_{xx} &= \mathbf{e}_x\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_x,\ T_{xy} = \mathbf{e}_x\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_y, \cdots,\ T_{zy} = \mathbf{e}_z\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_y,\ T_{zz} = \mathbf{e}_z\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_z. &(c)\\ \widehat{\mathbf{T}} &= (\mathbf{e}_r\mathbf{e}_r + \mathbf{e}_\theta\mathbf{e}_\theta + \mathbf{e}_\phi\mathbf{e}_\phi)\cdot\widehat{\mathbf{T}}\cdot(\mathbf{e}_r\mathbf{e}_r + \mathbf{e}_\theta\mathbf{e}_\theta + \mathbf{e}_\phi\mathbf{e}_\phi) &(d)\\ &= T_{rr}\mathbf{e}_r\mathbf{e}_r + T_{r\theta}\mathbf{e}_r\mathbf{e}_\theta + T_{r\phi}\mathbf{e}_r\mathbf{e}_\phi + \cdots + T_{\phi\theta}\mathbf{e}_\phi\mathbf{e}_\theta + T_{\phi\phi}\mathbf{e}_\phi\mathbf{e}_\phi, \quad \text{where} &(e)\\ T_{rr} &= \mathbf{e}_r\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_r,\ T_{r\theta} = \mathbf{e}_r\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_\theta, \cdots,\ T_{\phi\theta} = \mathbf{e}_\phi\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_\theta,\ T_{\phi\phi} = \mathbf{e}_\phi\cdot\widehat{\mathbf{T}}\cdot\mathbf{e}_\phi. &(f)\end{aligned} \tag{33}$$

[3] In QM the completeness of a set of orthonormal vectors $\{|u_i>; i = 1, 2, \ldots, \infty\}$ is expressed through the statement $\sum_i |u_i><u_i| = 1$. This relation is used to change the representation of a Hermitean operator $\widehat{\mathbf{T}}$, the equivalent of the tensors we are con-

Lines (a)-(c) represent the tensor $\widehat{\mathbf{T}}$ in a Cartesian coordinate system, and lines (d)-(f) in a spherical coordinate system

We can then write the components of $\widehat{\mathbf{T}}$ in the following matrix forms

$$\begin{array}{l} \widehat{\mathbf{T}} \overset{(\text{Cart})}{\longrightarrow} \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}. \\ \widehat{\mathbf{T}} \overset{(\text{sphr})}{\longrightarrow} \begin{pmatrix} T_{rr} & T_{r\theta} & T_{r\phi} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta\phi} \\ T_{\phi r} & T_{\phi\theta} & T_{\phi\phi} \end{pmatrix}. \end{array} \tag{34}$$

The first matrix gives the Cartesian components, and the second one the spherical components.

Using the transformation of the base vectors (Eq. 28), and the completeness relations (29), one can transform the Cartesian components to spherical components, for both vectors and tensors, as we shall show. For this purpose we shall temporarily denote the spherical base vectors with a prime, i.e., $\{\mathbf{e}'_i : i = r, \theta, \phi\}$ and make a table of *transformation coefficients* $\{c_{ij}\}$:

$$\begin{aligned} \mathbf{e}'_i &= \mathbf{e}'_i \cdot \mathbf{e}_j \mathbf{e}_j = c_{ij}\mathbf{e}_j, \\ \text{where} \quad c_{ij} &\equiv \mathbf{e}'_i \cdot \mathbf{e}_j : \; i = r, \theta, \phi; \; j = x, y, z. \\ &= \begin{pmatrix} \sin\theta\,\cos\phi & \sin\theta\,\sin\phi & \cos\theta \\ \cos\theta\,\cos\phi & \cos\theta\,\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \end{aligned} \tag{35}$$

Now, let $\mathbf{V}$ be a vector and $\widehat{\mathbf{T}}$ a tensor with Cartesian components $[\{V_j\},\ \{T_{ij}\},\ i, j = x, y, z]$ respectively. Then the spherical components of the same vector and tensor, namely, $[\{V'_j\},\ \{T'_{ij}\},\ i, j = r, \theta, \phi]$ will be obtained in the following ways[4] :

$$\begin{aligned} V'_j &= \mathbf{V} \cdot \mathbf{e}'_j = \mathbf{V} \cdot \mathbf{e}_k \mathbf{e}_k \cdot \mathbf{e}'_j = c_{jk} V_k. \\ T'_{ij} &= \mathbf{e}'_i \cdot \widehat{\mathbf{T}} \cdot \mathbf{e}'_j = \mathbf{e}'_i \cdot \mathbf{e}_k \mathbf{e}_k \cdot \widehat{\mathbf{T}} \cdot \mathbf{e}_l \mathbf{e}_l \cdot \mathbf{e}'_j \\ &= c_{ik} c_{jl} T_{kl}. \end{aligned} \tag{36}$$

Note that we have used the summation convention: sum over k in line (a), sum over k, l in line (b).

We shall illustrate the transformation formulas (36) with two examples, i.e,$V_r \equiv V'_1$ and $T_{r\theta} \equiv T'_{12}$.

$$\begin{aligned} &V_r = \sin\theta\cos\phi\, V_x + \sin\theta\sin\phi\, V_y + \cos\theta\, V_z. \\ &T_{r\theta} = \sin\theta\cos\phi\,(\cos\theta\cos\phi\, T_{xx} \\ &+ \cos\theta\sin\phi\, T_{xy} - \sin\theta\, T_{xz}) \\ &+ \sin\theta\sin\phi\,(\cos\theta\cos\phi\, T_{yx} + \cos\theta\sin\phi\, T_{yy} \\ &- \sin\theta\, T_{yz}) \\ &+ \cos\theta\,(\cos\theta\cos\phi\, T_{zx} + \cos\theta\sin\phi\, T_{zy} \\ &- \sin\theta\, T_{zz}). \end{aligned} \tag{37}$$

2.4 Example: Inertia Tensor

We shall illustrate the tensor concept by showing two important examples, namely (1)

[4] In Tensor analysis, the primary language of the theory of relativity, the rule of transformation has different forms for contravariant and covariant vectors, and for contravariant, covariant and mixed tensors. The rules we are establishing here are different from them. The components of vectors, tensors we are using may be called *physical components*, in contrast to their contravariant and covariant components for which a more elegant transformation rule is used.

the inertia tensor and (2) the stress tensor. We shall take up a short discussion of the first example in this section leaving the second example, which needs a more detailed coverage, to the next section.

In Sec. 2.1 we talked about the tensor operation converting the angular velocity $\boldsymbol{\omega}$ into angular momentum $\mathbf{L}$. The corresponding operator is the *inertia tensor* $\widehat{\boldsymbol{\mathcal{I}}}$ of the rigid body. Its dot product with the angular velocity $\boldsymbol{\omega}$ gives the *angular momentum* $\mathbf{L}$ of the rigid body. That is,

$$\mathbf{L} = \widehat{\boldsymbol{\mathcal{I}}} \cdot \boldsymbol{\omega}. \tag{38}$$

We shall find an expression for the vector angular momentum $\mathbf{L}$ of a rigid body which is rotating about a point O (which can be a moving point, e.g., the CM) with angular velocity $\boldsymbol{\omega} = \omega\mathbf{n}$ about the axis pointing in the direction of the unit vector $\mathbf{n}$. Let j be one of the constituent particles, having mass m_j, and located at the radius vector $\mathbf{r}_j$ with respect to O, as shown in Fig. 2(c). The velocity of this point is $\mathbf{v}_j = \boldsymbol{\omega} \times \mathbf{r}_j$. Therefore this particle has an angular momentum with respect to the point O, equal to

$$\begin{aligned}\boldsymbol{\ell}_j = \mathbf{r}_j \times \mathbf{p}_j = \mathbf{r}_j \times m_j\mathbf{v}_j = m_j\mathbf{r}_j \times (\boldsymbol{\omega} \times \mathbf{r}_j) \\ = m_j[r_j^2\boldsymbol{\omega} - (\mathbf{r}_j \cdot \boldsymbol{\omega})\mathbf{r}_j].\end{aligned} \tag{39}$$

Assuming that the rigid body is made of N particles (which is a very large number), we add the angular momentum of each particle to obtain the angular momentum of the rigid body about the point O, given as

$$\mathbf{L}_O = \sum_{j=0}^{N} m_j[r_j^2\boldsymbol{\omega} - (\mathbf{r}_j \cdot \boldsymbol{\omega})\mathbf{r}_j]. \tag{40}$$

We can write the quantity within square brackets as

$$[r_j^2\boldsymbol{\omega} - (\mathbf{r}_j \cdot \boldsymbol{\omega})\mathbf{r}_j] = [r_j^2\widehat{\mathbf{1}} - \mathbf{r}_j\mathbf{r}_j] \cdot \boldsymbol{\omega}, \tag{41}$$

and construct the Inertia tensor as the dyadic (sum of infinitely small dyads)

$$\widehat{\boldsymbol{\mathcal{I}}} = \sum_{j=0}^{N} m_j[r_j^2\widehat{\mathbf{1}} - \mathbf{r}_j\mathbf{r}_j]. \tag{42}$$

Then we get the angular momentum as the dot product

$$\mathbf{L}_O = \widehat{\boldsymbol{\mathcal{I}}} \cdot \boldsymbol{\omega}. \tag{43}$$

We have thus derived Eq. (38), and along with it have found an expression for the inertia tensor in Eq. (42). Note that the expression within the square brackets is the difference of two dyadics, namely, the identity dyadic $\widehat{\mathbf{1}}$ multiplied by the scalar r_j^2, and the dyadic product of $\mathbf{r}_j$ with itself.

For further clarification we shall write down the components of the tensor. Assumimg that the rigid body has uniform mass density ρ distributed over its volume V, the sum in Eq. (42) becomes the integral:

$$\widehat{\boldsymbol{\mathcal{I}}} = \rho \iiint_V [r^2\widehat{\mathbf{1}} - \mathbf{r}\mathbf{r}]d^3r. \tag{44}$$

Some of its components are

$$\begin{aligned}\mathcal{I}_{xx} &= \rho \iiint_V [r^2 - x^2]d^3r = \rho \iiint_V (y^2 + z^2)d^3r; \\ \mathcal{I}_{xy} &= -\rho \iiint_V (xy)\, d^3r; \quad \text{etc.}\end{aligned} \tag{45}$$

It is now seen that the *inertia tensor is a symmetric tensor*, i.e.,

$$\mathcal{I}_{xy} = \mathcal{I}_{yx}; \quad \mathcal{I}_{yz} = \mathcal{I}_{zy}; \quad \mathcal{I}_{zx} = \mathcal{I}_{xz}. \quad (46)$$

This symmetry property is preserved under all coordinate transformations.

3 Stress in a Medium

Stress and Stress Tensor are discussed in engineering books on Fluid Mechanics[10] and Strength of Materials[11]. However, physics students may find useful the introductory lessons on these concepts by Symon[12] and Feynman[13].

3.1 Stress Vector

By (mechanical) stress we mean *internal forces* (in the form of intermolecular interactions) called into play when bulk matter, either in the form of solid, liquid or gas, is subjected to external forces. These internal forces exist throughout the bulk matter and its mathematical expression is given by a stress tensor field $\widehat{\boldsymbol{\mathcal{T}}}(x, y, z)$.

For simplicity we shall consider a solid block in Fig. 3(a). It has been cut into two parts, the upper block $\mathcal{U}$ and the lower block $\mathcal{L}$, by an imaginary plane Σ, leaving a trace Γ of its boundary. This plane is identified by the unit normal vector $\mathbf{n}$ pointing from the lower block to the upper block.

In Fig. 3(b) we have shown the lower block $\mathcal{L}$ with the plane of separation Σ exposed. Let us consider a small area da at the point $\mathrm{P}(x, y, z)$ inside the solid, but lying on this plane. Then the *stress vector* $\boldsymbol{\mathcal{T}}^{(n)}(x, y, z)$ is defined to be the force per unit area at $\mathrm{P}(x, y, z)$, exerted by the atoms of the upper block $\mathcal{U}$ on the atoms of the lower block $\mathcal{L}$ across the plane $\mathbf{n}$. The infinitesimal force acting on the area da is then

$$d\mathbf{F}^{(n)} = \boldsymbol{\mathcal{T}}^{(n)}(x, y, z)\, da. \quad (47)$$

Note that in general the direction of the stress vector $\boldsymbol{\mathcal{T}}^{(n)}(x, y, z)$ is different from the direction of the normal $\mathbf{n}$. If, however, $\boldsymbol{\mathcal{T}}^{(n)}(x, y, z) \parallel \mathbf{n}$ (i.e., perpendicular to the plane), the stress (vector) is called *normal stress*. If $\boldsymbol{\mathcal{T}}^{(n)}(x, y, z) \perp \mathbf{n}$ (i.e., parallel to the plane), it is called *shear stress*.

3.2 Stress Tensor

In Fig. 3(c) we have shown the stress vectors $\boldsymbol{\mathcal{T}}^{(x)}, \boldsymbol{\mathcal{T}}^{(y)}, \boldsymbol{\mathcal{T}}^{(z)}$ on three perpendicular faces of a tiny rectangular block, identified by the normal vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$. Let $\{n_x, n_y, n_z\}$ be the direction cosines of $\mathbf{n}$ so that

$$\mathbf{n} = n_x\mathbf{e}_x + n_x\mathbf{e}_x + n_x\mathbf{e}_z. \quad (48)$$

It can be shown, using the equation of motion of the prism shown in Fig. 3(d) that

$$\boldsymbol{\mathcal{T}}^{(n)} = \boldsymbol{\mathcal{T}}^{(x)}n_x + \boldsymbol{\mathcal{T}}^{(y)}n_y + \boldsymbol{\mathcal{T}}^{(z)}n_z. \quad (49)$$

Eq. (49) can be given an elegant form if we write the stress vectors as column matrices

$$\boldsymbol{\mathcal{T}}^{(n)} = \begin{pmatrix} \mathcal{T}_x^{(n)} \\ \mathcal{T}_y^{(n)} \\ \mathcal{T}_z^{(n)} \end{pmatrix}; \; \boldsymbol{\mathcal{T}}^{(x)} = \begin{pmatrix} \mathcal{T}_{xx} \\ \mathcal{T}_{yx} \\ \mathcal{T}_{zx} \end{pmatrix};$$
$$\boldsymbol{\mathcal{T}}^{(y)} = \begin{pmatrix} \mathcal{T}_{xy} \\ \mathcal{T}_{yy} \\ \mathcal{T}_{zy} \end{pmatrix}; \; \boldsymbol{\mathcal{T}}^{(z)} = \begin{pmatrix} \mathcal{T}_{xz} \\ \mathcal{T}_{yz} \\ \mathcal{T}_{zz} \end{pmatrix}; \quad (50)$$

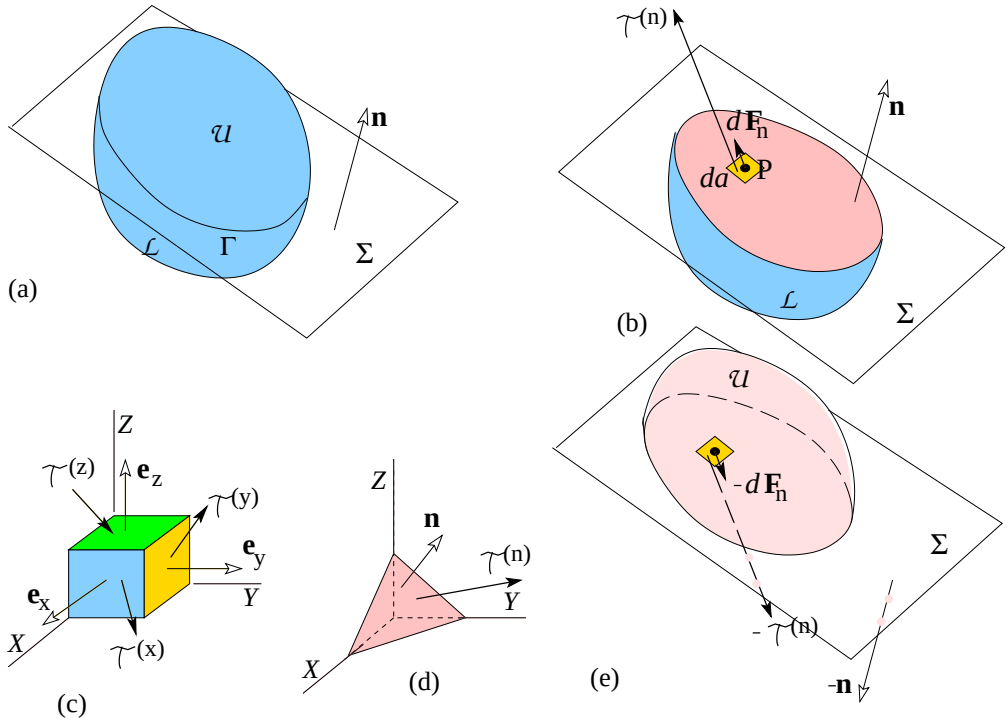

Figure 3: *Explaining the Stress Tensor*

invoke a *Stress Tensor* $\widehat{\boldsymbol{\mathcal{T}}}$ having the *matrix representation*

$$\widehat{\boldsymbol{\mathcal{T}}} = \begin{pmatrix} \mathcal{T}_{xx} & \mathcal{T}_{xy} & \mathcal{T}_{xz} \\ \mathcal{T}_{yx} & \mathcal{T}_{yy} & \mathcal{T}_{yz} \\ \mathcal{T}_{zx} & \mathcal{T}_{zy} & \mathcal{T}_{zz} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mathcal{T}}^{(x)} & \boldsymbol{\mathcal{T}}^{(y)} & \boldsymbol{\mathcal{T}}^{(z)} \end{pmatrix} \tag{51}$$

so that Eq. (49) can be represented by the following matrix multiplication.

$$\begin{pmatrix} \mathcal{T}_x^{(n)} \\ \mathcal{T}_y^{(n)} \\ \mathcal{T}_z^{(n)} \end{pmatrix} = \begin{pmatrix} \mathcal{T}_{xx} & \mathcal{T}_{xy} & \mathcal{T}_{xz} \\ \mathcal{T}_{yx} & \mathcal{T}_{yy} & \mathcal{T}_{yz} \\ \mathcal{T}_{zx} & \mathcal{T}_{zy} & \mathcal{T}_{zz} \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}. \tag{52}$$

Alternatively, we can write the stress tensor in the *dyadic representation*

$$\widehat{\boldsymbol{\mathcal{T}}} = \boldsymbol{\mathcal{T}}^{(x)}\mathbf{e}_x + \boldsymbol{\mathcal{T}}^{(y)}\mathbf{e}_y + \boldsymbol{\mathcal{T}}^{(z)}\mathbf{e}_z, \tag{53}$$

so that Eq. (49) can be retrieved from the dot product of the above dyadic with the unit vector $\mathbf{n}$ placed on the right, i.e.,.

$$\boldsymbol{\mathcal{T}}^{(n)} = \widehat{\boldsymbol{\mathcal{T}}} \cdot \mathbf{n}. \tag{54}$$

Note from (51) that in $\mathcal{T}_{ij}$ the second index j is the "surface index" (indicating the direction of the surface on which stands the stress vector $\boldsymbol{\mathcal{T}}^{(j)}$) and the first index i the "component index" (indicating x, y, z component of $\boldsymbol{\mathcal{T}}^{(j)}$).

In Fig. 3(e) we have shown the upper part of the solid of Fig.(a), and the same area da as in Fig.(b), but now on the upper block $\mathcal{U}$. The normal vector now is $-\mathbf{n}$, and the stress vector is

$$\begin{aligned} \boldsymbol{\mathcal{T}}^{(-n)}(x,y,z) &= \widehat{\boldsymbol{\mathcal{T}}}(x,y,z) \cdot (-\mathbf{n}) \\ &= -\boldsymbol{\mathcal{T}}^{(n)}(x,y,z), \end{aligned} \tag{55}$$

so that the force exerted by the atoms of the lower block $\mathcal{L}$ on the atoms of the upper block $\mathcal{U}$ across the same area da is $d\mathbf{F}'^{(n)} = -\boldsymbol{\mathcal{T}}^{(n)}da = -d\mathbf{F}^{(n)}$. Which is in conformity with Newton's Third Law of Motion.

In obtaining the last equality in Eq. (55) we have used the linearity property of the tensor as stipulated in (3). In this case $\widehat{\boldsymbol{\mathcal{T}}} \cdot (a\mathbf{n}) = a\widehat{\boldsymbol{\mathcal{T}}} \cdot \mathbf{n}$ where $a = -1$.

Like the inertia tensor, the stress tensor is a *symmetric tensor*, i.e.,

$$\mathcal{T}_{xy} = \mathcal{T}_{yx}; \quad \mathcal{T}_{yz} = \mathcal{T}_{zy}; \quad \mathcal{T}_{zx} = \mathcal{T}_{xz}. \tag{56}$$

which can be proved using the equation of motion of the angular momentum.

3.3 Gauss's Divergence Theorem for a Tensor Field

When we say tensor field, we mean a physical quantity represented by a tensor $\widehat{\mathbf{T}}(x,y,z)$ whose nine components $T_{xx}(x,y,z), T_{xy}(x,y,z), \cdots T_{zz}(x,y,z)$ are defined at every coordinate point (x,y,z). We assume that these nine components are all differentiable functions of the coordinates x, y, z. For such a tensor field we define its *divergence* to be the formal dot product of the grad operator $\boldsymbol{\nabla}$ with the tensor $\widehat{\mathbf{T}}(x,y,z)$, it being assumed that $\boldsymbol{\nabla}$ will appear on the left.

Let us write the tensor $\widehat{\mathbf{T}}$ by the dyadic representation

$$\widehat{\mathbf{T}} = \mathbf{T}^{(x)}\mathbf{e}_x + \mathbf{T}^{(y)}\mathbf{e}_y + \mathbf{T}^{(z)}\mathbf{e}_z, \tag{57}$$

as in Eq. (53). Then

$$\operatorname{div}\widehat{\mathbf{T}} \equiv \boldsymbol{\nabla}\cdot\widehat{\mathbf{T}} = \boldsymbol{\nabla}\cdot(\mathbf{T}^{(x)}\mathbf{e}_x + \mathbf{T}^{(y)}\mathbf{e}_y + \mathbf{T}^{(z)}\mathbf{e}_z) \stackrel{\text{def}}{=} \left(\boldsymbol{\nabla}\cdot\mathbf{T}^{(x)}\right)\mathbf{e}_x + \left(\boldsymbol{\nabla}\cdot\mathbf{T}^{(y)}\right)\mathbf{e}_y + \left(\boldsymbol{\nabla}\cdot\mathbf{T}^{(z)}\right)\mathbf{e}_z. \tag{58}$$

Note that $\boldsymbol{\nabla}\cdot\mathbf{T}^{(x)}, \boldsymbol{\nabla}\cdot\mathbf{T}^{(y)}, \boldsymbol{\nabla}\cdot\mathbf{T}^{(z)}$ are the familiar scalar divergences of the vector fields $\mathbf{T}^{(x)}, \mathbf{T}^{(y)}, \mathbf{T}^{(z)}$ respectively,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{T}^{(x)} &= \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z},\\
\boldsymbol{\nabla}\cdot\mathbf{T}^{(y)} &= \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zx}}{\partial z},\\
\boldsymbol{\nabla}\cdot\mathbf{T}^{(z)} &= \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z}.
\end{aligned} \tag{59}$$

and constitute three (scalar) components of the vector $\boldsymbol{\nabla}\cdot\widehat{\mathbf{T}}$ along the X, Y and Z axes respectively. Combining (58) and (59) we get

$$\boldsymbol{\nabla}\cdot\widehat{\mathbf{T}} = \sum_{j=1}^{3}\sum_{i=1}^{3}\frac{\partial T_{ij}}{\partial x_i}\,\mathbf{e}_j \equiv \frac{\partial T_{ij}}{\partial x_i}\,\mathbf{e}_j. \tag{60}$$

In the second equality we have employed Einestein's summation convention (introduced on page 8.)

The divergence of a vector field is sometimes interpreted as "outflux per unit volume". This association of divergence with outflux is due to Gauss's divergence theorem. Applying the divergence theorem to the three vector fields $\mathbf{T}^{(x)}, \mathbf{T}^{(y)}, \mathbf{T}^{(z)}$ separately, we get the following three equivalence relations.

$$\begin{aligned}
\iiint_V \boldsymbol{\nabla}\cdot\mathbf{T}^{(x)}(\mathbf{r})\,d^3r &= \iint_S \mathbf{n}(\mathbf{r})\cdot\mathbf{T}^{(x)}(\mathbf{r})\,da.\\
\iiint_V \boldsymbol{\nabla}\cdot\mathbf{T}^{(y)}(\mathbf{r})\,d^3r &= \iint_S \mathbf{n}(\mathbf{r})\cdot\mathbf{T}^{(y)}(\mathbf{r})\,da.\\
\iiint_V \boldsymbol{\nabla}\cdot\mathbf{T}^{(z)}(\mathbf{r})\,d^3r &= \iint_S \mathbf{n}(\mathbf{r})\cdot\mathbf{T}^{(z)}(\mathbf{r})\,da.
\end{aligned} \tag{61}$$

Multiplying either side of the first, second and third lines with $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ respectively, and adding, we get

$$\begin{aligned}
&\iiint_V \boldsymbol{\nabla}\cdot\left(\mathbf{T}^{(x)}\mathbf{e}_x + \mathbf{T}^{(y)}\mathbf{e}_y + \mathbf{T}^{(z)}\mathbf{e}_z\right), d^3r\\
&= \iint_S \mathbf{n}(\mathbf{r})\cdot\left(\mathbf{T}^{(x)}\mathbf{e}_x + \mathbf{T}^{(y)}\mathbf{e}_y + \mathbf{T}^{(z)}\mathbf{e}_z\right)\,da.
\end{aligned} \tag{62}$$

Identifying the dyadic within the paranthe-ses as the tensor $\widehat{\mathbf{T}}$, we obtain the divergence theorem for the tensor field.

$$\iiint_V \boldsymbol{\nabla}\cdot\widehat{\mathbf{T}}(\mathbf{r})\,d^3r = \iint_S \mathbf{n}(\mathbf{r})\cdot\widehat{\mathbf{T}}(\mathbf{r})\,da. \tag{63}$$

Specializing the above theorem to stress tensor, using its symmetry property, we can write the integrand on the right side as

$$\begin{aligned}
\mathbf{n}\cdot\widehat{\boldsymbol{\mathcal{T}}} &= n_j\boldsymbol{\mathcal{T}}_{jk}\mathbf{e}_k = \mathbf{e}_k\boldsymbol{\mathcal{T}}_{kj}n_j\\
&= \mathbf{e}_k(\boldsymbol{\mathcal{T}}\cdot\mathbf{n})_k = \widehat{\boldsymbol{\mathcal{T}}}\cdot\mathbf{n}.
\end{aligned} \tag{64}$$

We shall write the divergence theorem for stress tensor in the following form

$$\boxed{\iiint_V \boldsymbol{\nabla}\cdot\widehat{\boldsymbol{\mathcal{T}}}(\mathbf{r})\,d^3r = \iint_S \cdot\widehat{\boldsymbol{\mathcal{T}}}(\mathbf{r})\cdot\mathbf{n}\,da.} \tag{65}$$

We shall find Eq. (65) to be crucial for constructing Maxwell's stress tensor in the following sections.

3.4 Volume force density in a stress tensor field

Fig. 4 shows an imaginary rectangular box *abcdefgh* of infinitesimal dimensions

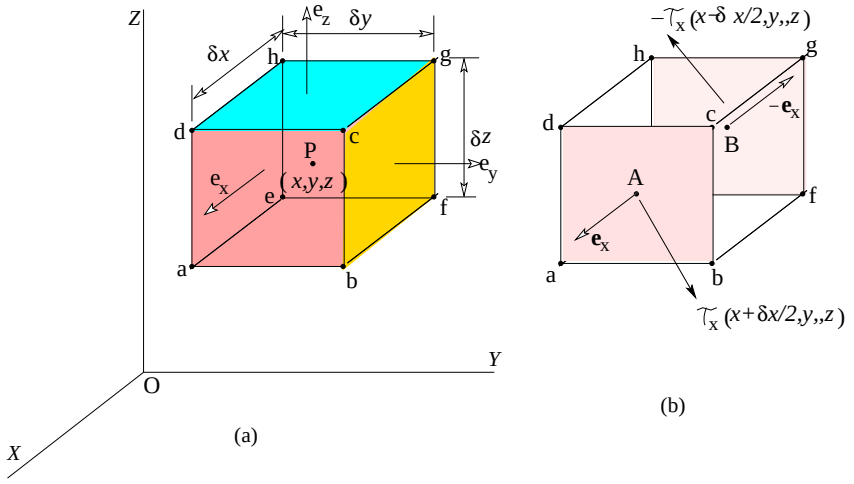

Figure 4: *Stress Force on a Volume Element*

$\delta x, \delta y, \delta z$ inside a medium under stress (which may be matter, or field). The centre P of this box is located at the coordinates (x, y, z). Let us assume that the stress in the medium is given by the tensor field $\widehat{\boldsymbol{\mathcal{T}}}(x, y, z)$, whose components are differentiable functions of the coordinates. We shall find the total force on this box due to this stress.

We have shown in Fig.(a) the outward normal vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ on the three faces of the box that are exposed to our view. The outward normals on the other faces which are hidden from our view are $(-\mathbf{e}_x, -\mathbf{e}_y, -\mathbf{e}_z)$. We shall identify each one of the six surfaces of the box by their outward normal vectors.

Let us consider the opposite faces *abcd* and *efgh*, recognized by the normals $(\mathbf{e}_x)$ and $(-\mathbf{e}_x)$. The locations of their centres are $(x + \frac{\delta x}{2}, y, z)$ and $(x - \frac{\delta x}{2}, y, z)$ respectively. The stress forces on these two faces are

$$\begin{aligned}
\delta\mathbf{F}_{+x} &= \widehat{\boldsymbol{\mathcal{T}}}(x + \tfrac{\delta x}{2}, y, z) \cdot (+\mathbf{e}_x)\, \delta y\, \delta z \\
&= \boldsymbol{\mathcal{T}}^{(x)}(x + \tfrac{\delta x}{2}, y, z)\, \delta y\, \delta z \\
&= [\boldsymbol{\mathcal{T}}^{(x)}(x, y, z) + \tfrac{\partial \boldsymbol{\mathcal{T}}^{(x)}}{\partial x} \tfrac{\delta x}{2}]\, \delta y\, \delta z \\
\delta\mathbf{F}_{-x} &= \widehat{\boldsymbol{\mathcal{T}}}(x - \tfrac{\delta x}{2}, y, z) \cdot (-\mathbf{e}_x)\, \delta y\, \delta z \\
&= -\boldsymbol{\mathcal{T}}^{(x)}(x - \tfrac{\delta x}{2}, y, z)\, \delta y\, \delta z \\
&= -[\boldsymbol{\mathcal{T}}^{(x)}(x, y, z) - \tfrac{\partial \boldsymbol{\mathcal{T}}^{(x)}}{\partial x} \tfrac{\delta x}{2}]\, \delta y\, \delta z
\end{aligned}$$

$$\delta\mathbf{F}_{+x} + \delta\mathbf{F}_{-x} = \frac{\partial \boldsymbol{\mathcal{T}}^{(x)}}{\partial x} \delta x\, \delta y\, \delta z = \frac{\partial \boldsymbol{\mathcal{T}}^{(x)}}{\partial x} \delta V. \tag{66}$$

where $\delta V = \delta x\, \delta y\, \delta z$ is the volume of the infinitesimal box. In the same way we find the forces on the other four faces of the block. Adding the stress forces on all the six surfaces we get

$$\delta\mathbf{F}_s = \left[\frac{\partial \boldsymbol{\mathcal{T}}^{(x)}}{\partial x} + \frac{\partial \boldsymbol{\mathcal{T}}^{(y)}}{\partial y} + \frac{\partial \boldsymbol{\mathcal{T}}^{(z)}}{\partial z} \right] \delta V \tag{67}$$

as the total stress force on the box. The volume force density $\mathbf{f}_s$, which gives the stress

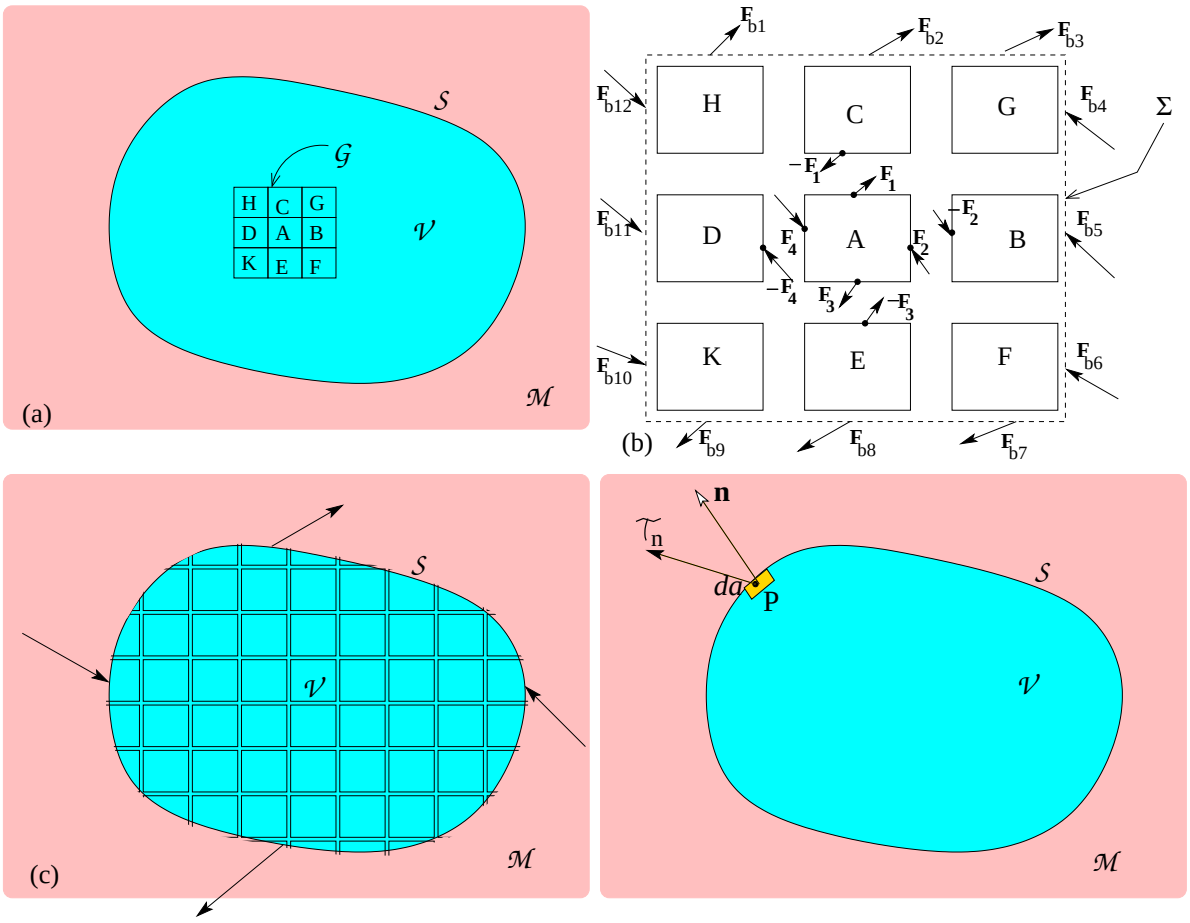

Figure 5: *Stress Forces on a Bulk Volume*

force acting per unit volume of the media under stress, is then given as

$$\begin{aligned}\mathbf{f}_s &= \frac{\partial \boldsymbol{\mathcal{T}}^{(x)}}{\partial x} + \frac{\partial \boldsymbol{\mathcal{T}}^{(y)}}{\partial y} + \frac{\partial \boldsymbol{\mathcal{T}}^{(z)}}{\partial z} \\ &= \frac{\partial (\mathbf{e}_x \cdot \widehat{\boldsymbol{\mathcal{T}}})}{\partial x} + \frac{\partial (\mathbf{e}_y \cdot \widehat{\boldsymbol{\mathcal{T}}})}{\partial y} + \frac{\partial (\mathbf{e}_z \cdot \widehat{\boldsymbol{\mathcal{T}}})}{\partial z} \\ &= \left[\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_x \frac{\partial}{\partial x}\right] \cdot \widehat{\boldsymbol{\mathcal{T}}} \\ &= \boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}.\end{aligned} \tag{68}$$

One may conclude that total stress force $\mathbf{F}_s$ on a bulk volume $\mathcal{V}$ carved out inside a medium $\mathcal{M}$, as shown in Fig.5(a), is the volume integral of the force density $\mathbf{f}_s$ carried out over the entire volume $\mathcal{V}$. We shall carefully analyze the forces inside the medium before jumping into this conclusion.

Let us consider a two-diemnsional view of nine tiny, imaginary neighbouring blocks lying inside the medium and forming a group $\mathcal{G}$. We have marked the blocks as A,B,C,D,E,F,G,H,K, with A at the centre. In Fig.(b) we have shown the forces on the four sides of A as $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4$. The force $\mathbf{F}_1$ comes from the neighbour B, and by Newton's third law of motion, A applies an equal and opposite force $-\mathbf{F}_1$ on B. Similarly, the forces $\mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4$ come from the neighbours C, D, E . And A applies equal and opposite forces $-\mathbf{F}_2, -\mathbf{F}_3, -\mathbf{F}_4$ on them. It *may* then appear that these internal forces, when added together, get cancelled out and there should not be any stress force on the group $\mathcal{G}$ at all.

A close examination will disprove this judgement. We have surrounded $\mathcal{G}$ by an imaginary boundary surface Σ. It is now seen that even though the action-reaction forces cancel out in the interior of the group $\mathcal{G}$, they survive on the boundary surface Σ. These surface forces $\mathbf{F}_{b1}, \mathbf{F}_{b2}, \cdots \mathbf{F}_{b12}$, when added together constitute the total force $\mathbf{F}_s$ on the group $\mathcal{G}$.

In Fig.(c) we have divided the volume $\mathcal{V}$ into an infinite number of infinitesimal blocks. The interior stress forces between adjoining blocks will cancel out. However, the forces on the bounadry surface, some of which we have shown as $\mathbf{F}_{b1}, \mathbf{F}_{b2}, \mathbf{F}_{b3}, \mathbf{F}_{b4}$, will survive and add together to constitute the net stress force $\mathbf{F}_s$ on the volume $\mathcal{V}$.

We now get a clue of how to find the net stress force $\mathbf{F}_s$ on the volume $\mathcal{V}$. In Fig.(d) we have shown the volume $\mathcal{V}$ once again. At a certain point P on this surface we have pictured a tiny patch of area da, on which we have drawn a unit outward normal $\mathbf{n}$. The stress force on this patch is $d\mathbf{f}_s = \boldsymbol{\mathcal{T}}^{(n)}\, da = \widehat{\boldsymbol{\mathcal{T}}} \cdot \mathbf{n}\, da$. Integrate this force over the entire boundary to get $\mathbf{F}_s$. We shall perform this integration and convert the surface integral into volume integral by applying Gauss's Divergence Theorem as derived in Eq. (63).

$$\mathbf{F}_s = \iint_S \widehat{\boldsymbol{\mathcal{T}}}(\mathbf{r}) \cdot \mathbf{n}\, da = \iiint_V \boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}(\mathbf{r})\, d^3 r. \tag{69}$$

We have thus confirmed our guess following Eq. (68). We shall rewrite the same equation with emphasis, as this equation will serve as the cornerstone for the constructon of Maxwell's Stress tensor.

$$\boxed{\mathbf{f}_s(\mathbf{r}) = \boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}(\mathbf{r}).} \tag{70}$$

4 Maxwell's Stress Tensor for the Electrostatic Field

4.1 Volume force density in terms of the field

We shall now construct the stress tensor for the electrostatic filed. We shall call this tensor *Maxwell's Stress Tensor* and represent it by the symbol $\widehat{\boldsymbol{\mathcal{T}}}^{(\text{E})}$, where the superscript $^{(\text{E})}$ implies Electric field.

Fig. 6 shows a system of electric charges $\mathcal{S}$ placed in an Electric field $\mathbf{E}(\mathbf{r})$. In Fig.(a) the system consists of discrete charges $q_1, q_2, q_3, \cdots$ placed at the radius vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \cdots$. In Fig.(b) the system is a continuous distribution characterized by a smooth charge density function $\rho(\mathbf{r})$ confined within a volume. Our intention is to write the total electric force $\mathbf{F}$ on this system.

The force on the discrete system shown in Fig.(a) is given as

$$\mathbf{F} = \sum_j q_j \mathbf{E}^{(\text{ext})}(\mathbf{r}_j). \tag{71}$$

Here the sum is over all the charges in the system, and $\mathbf{E}^{(\text{ext})}(\mathbf{r}_j)$ is the *external electric field* at the radius vector $\mathbf{r}_j$ caused by the presence of all *other* charges lying outside the system $\mathcal{S}$.

For the case of continuous distribution, shown in Fig.(b), the individual charges become infinitesimal elementary charges i.e., $q_j \to \rho(\mathbf{r}) d^3r$, and the sum becomes the integral

$$\mathbf{F} = \iiint_V \rho(\mathbf{r}) \mathbf{E}^{(\text{ext})}(\mathbf{r})\, d^3r. \tag{72}$$

What about the force from the charges inside the system $\mathcal{S}$. They are *internal* forces, and cancel due to Newton's third law of motion.

Let $\mathbf{E}_i^{(\text{int})}(\mathbf{r}_j)$ be the "internal" field caused at $\mathbf{r}_j$ by a member particle i lying within the sytem $\mathcal{S}$. Then $\mathbf{F}_{ij} = q_j \mathbf{E}_i^{(\text{int})}(\mathbf{r}_j)$ is the force that the member particle i exerts on the member particle j. By Newton's third law of motion, $q_j \mathbf{E}_i^{(\text{int})}(\mathbf{r}_j) + q_i \mathbf{E}_j^{(\text{int})}(\mathbf{r}_i) = \mathbf{0}$. Adding together over all pairs for the discrete distribution, and integrating over the entire distribution for the continuous distribution we get

$$\begin{aligned} &\text{For discrete: } \textstyle\sum_{j=1}^{N} q_j \sum_{i=1}^{'N} \mathbf{E}_i^{(\text{int})}(\mathbf{r}_j) \\ &\quad = \textstyle\sum_{j=1}^{N} q_j \mathbf{E}^{(\text{int})}(\mathbf{r}_j) = \mathbf{0}. \\ &\text{For continuous: } \iiint_V \rho(\mathbf{r})\, \mathbf{E}^{(\text{int})}(\mathbf{r}) = \mathbf{0}. \end{aligned} \tag{73}$$

In the first equation the sum symbol $\sum'$ means that while summing over i, the term $i = j$ (corresponding to the "self field" of the member j) is to be avoided. The "internal field" $\mathbf{E}^{(\text{int})}(\mathbf{r}_j)$ is the field at the location of the member j caused by "all other members" in the system $\mathcal{S}$. In the second equation $\mathbf{E}^{(\text{int})}(\mathbf{r})$ is the "internal field" at the radius vector $\mathbf{r}$, as sensed by a tiny volume element d^3r at this point.

We shall add the null contribution shown in the second line of Eq. (73) to the right side of Eq. (72) and write

$$\mathbf{F} = \iiint_V \rho(\mathbf{r}) \mathbf{E}\,(\mathbf{r})\, d^3r. \tag{74}$$

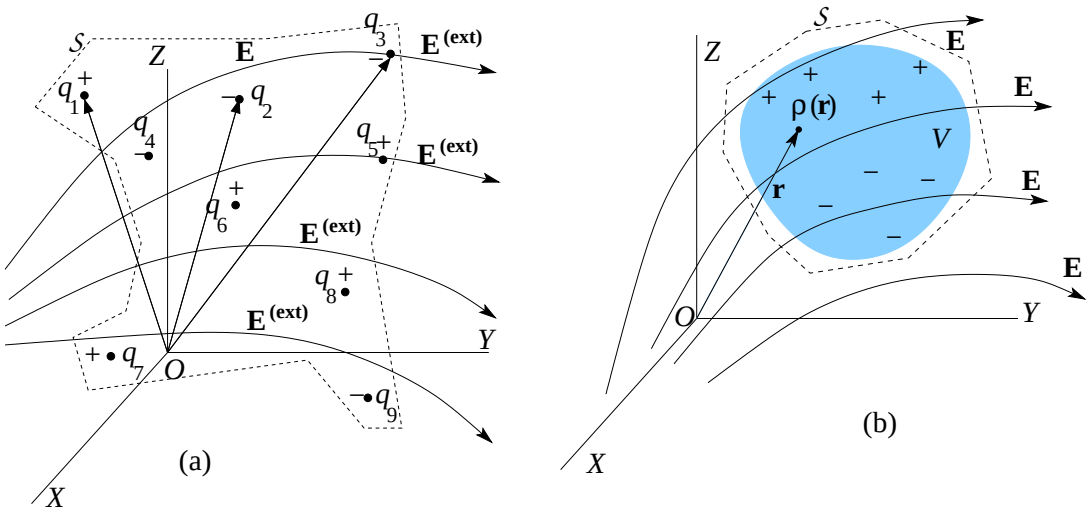

Figure 6: *Forces on Charges in an Electric Field*

Here $\mathbf{E}(\mathbf{r})$ is the actual field at the point $\mathbf{r}$, being the sum of two contributions, from the (i) external sources, and (ii) the internal sources of the system $\mathcal{S}$.

The purpose of adding the null integral of Eq. (73b) to Eq. (72) is that when we write the force density $\mathbf{f}$, the internal forces need to be added. That is,

$$\mathbf{f}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{E}(\mathbf{r}) \tag{75}$$

is the force on unit volume of the charge distribution at $\mathbf{r}$, in which $\mathbf{E}(\mathbf{r})$ is necessarily the *total* field at this location, caused by *both* external and internal sources. Now we manipulate the right hand side of Eq. (75) so as to convert $\rho\mathbf{E} \to \boldsymbol{\nabla}\cdot\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}$, as suggested in Eq. (70). This new tensor field $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}(\mathbf{r})$ would represent "stress" in the electrostatic field.

Construction of the stress tensor for electrostatic field, magnetostatic field and time varying electromagnetic field will be facilitated by the following identity[14]

$$\boldsymbol{\nabla}\cdot\left[\mathbf{AA} - \frac{1}{2}A^2\widehat{\mathbf{1}}\right] = (\boldsymbol{\nabla}\cdot\mathbf{A})\mathbf{A} - \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A}). \tag{76}$$

Before establishing the above identity we shall need a standard formula (See for example, *Vector Formulas* compiled in Griffiths, 3rd Ed)

$$\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}. \tag{77}$$

By setting $\mathbf{B} = \mathbf{A}$ in the above formula and get

$$\boldsymbol{\nabla}\left(\frac{1}{2}A^2\right) = \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{A}. \tag{78}$$

We shall now prove the identity (76).

Proof:

$$
\begin{aligned}
&\boldsymbol{\nabla}\cdot(\mathbf{AA}) = \left(\mathbf{e}_l\tfrac{\partial}{\partial x_l}\right)\cdot(\mathbf{e}_i\mathbf{e}_j A_i A_j)\\
&= \tfrac{\partial}{\partial x_i}(A_i A_j)\mathbf{e}_j\\
&= \left\{\left(\tfrac{\partial A_i}{\partial x_i}\right)A_j + \left(A_i\tfrac{\partial}{\partial x_i}\right)A_j\right\}\mathbf{e}_j\\
&= (\boldsymbol{\nabla}\cdot\mathbf{A})\mathbf{A} + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{A}. \qquad (a)\\
&\boldsymbol{\nabla}\cdot\left(\tfrac{1}{2}A^2\widehat{\mathbf{1}}\right) = \left(\mathbf{e}_l\tfrac{\partial}{\partial x_l}\right)\cdot\left(\tfrac{1}{2}\mathbf{e}_i\mathbf{e}_i A^2\right)\\
&= \tfrac{1}{2}\mathbf{e}_i\tfrac{\partial A^2}{\partial x_i} = \boldsymbol{\nabla}\left(\tfrac{1}{2}A^2\right)\\
&= \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{A}, \text{ by (78)}. \qquad (b)
\end{aligned}
$$

The identity (76) follows when we subtract line (b) from line (a).

Q.E.D.

Note that we have used Einstein's summation convention introduced on page 8. That is, $\mathbf{e}_l\frac{\partial}{\partial x_l} \equiv \sum_{l=1}^{3}\mathbf{e}_l\frac{\partial}{\partial x_l}$; $\mathbf{e}_i\mathbf{e}_j A_i A_j \equiv \sum_{i=1}^{3}\sum_{j=1}^{3}\mathbf{e}_i\mathbf{e}_j A_i A_j$, etc.

The stress tensor for the electrostatic field follows when we set $\mathbf{E}$ for $\mathbf{A}$ in (76), and use the field equations: $\boldsymbol{\nabla}\cdot\mathbf{E} = \rho/\epsilon_0$; $\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$:

$$
\boxed{\begin{aligned}
\mathbf{f}^{(\mathrm{E})} &= \rho\mathbf{E} = \boldsymbol{\nabla}\cdot\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}, \qquad (a)\\
\text{where } \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} &= \epsilon_0\left[\mathbf{EE} - \tfrac{1}{2}E^2\widehat{\mathbf{1}}\right]. \qquad (b)
\end{aligned}} \tag{79}
$$

It will be a simple exercise to write the Cartesian components of this tensor:

$$
\begin{aligned}
\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} &= \left(\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}\cdot\mathbf{e}_x \quad \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}\cdot\mathbf{e}_y \quad \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}\cdot\mathbf{e}_z\right)\\
&= \epsilon_0\begin{pmatrix} \frac{1}{2}(E_x^2 - E_y^2 - E_z^2) & E_xE_y & E_xE_z\\ E_yE_x & \frac{1}{2}(E_y^2 - E_z^2 - E_x^2) & E_yE_z\\ E_zE_x & E_zE_y & \frac{1}{2}(E_z^2 - E_y^2 - E_z^2)\end{pmatrix}.
\end{aligned} \tag{80}
$$

4.2 Example: Stress vector on a plane as a function of the angle of inclination

The stress tensor (79) will remain abstract and obscure unless the reader works out a few examples. Griffiths has shown a beautiful example: the force on the upper half of a uniformly charged sphere using the stress tensor as given in formula (79b). However, he has worked in the Cartesian coordinate system. The reader should work out the same problem using the spherical coordinate system, spending much less time in getting the answer.

We shall provide two examples of which the first one is depicted in Fig. 7. A *uniform* electric field $\mathbf{E} = E\mathbf{e}_x$ exists in a certain region of space. The stress tensor is then given by

the following expression.

$$\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} = \frac{\epsilon_0}{2}E^2(\mathbf{e}_x\mathbf{e}_x - \mathbf{e}_y\mathbf{e}_y - \mathbf{e}_z\mathbf{e}_z) = \frac{\epsilon_0}{2}\begin{pmatrix} E^2 & 0 & 0 \\ 0 & -E^2 & 0 \\ 0 & 0 & -E^2 \end{pmatrix}. \quad (81)$$

Imagine a plane running parallel to the Z axis, but inclined to the X axis by an angle θ (Fig a). The normal vector is then given as

$$\mathbf{n} = \mathbf{e}_x \sin\theta + \mathbf{e}_y \cos\theta = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix}. \quad (82)$$

The stress vector $\boldsymbol{\mathcal{T}}^{(n)}$ on this plane is then

$$\boldsymbol{\mathcal{T}}^{(n)} = \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} \cdot \mathbf{n} = \frac{\epsilon_0}{2}E^2(\mathbf{e}_x \sin\theta - \mathbf{e}_y \cos\theta) = \frac{\epsilon_0}{2}E^2\begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix}. \quad (83)$$

Let us consider some special cases.

$$\begin{aligned} \boldsymbol{\mathcal{T}}^{(x)} &= \tfrac{\epsilon_0}{2}E^2\mathbf{e}_x, \quad (\text{by setting } \theta = \pi/2) && (a) \\ \boldsymbol{\mathcal{T}}^{(y)} &= -\tfrac{\epsilon_0}{2}E^2\mathbf{e}_y, \quad (\text{by setting } \theta = 0) && (b) \\ \boldsymbol{\mathcal{T}}^{(z)} &= -\tfrac{\epsilon_0}{2}E^2\mathbf{e}_z, \quad (\text{same as } \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} \cdot \mathbf{e}_z) && (c) \\ \boldsymbol{\mathcal{T}}^{(45^o)} &= \tfrac{\epsilon_0}{2}E^2\tfrac{1}{\sqrt{2}}(\mathbf{e}_x - \mathbf{e}_y). && (d) \end{aligned} \quad (84)$$

Lines (a) - (c) give the stress vectors on the planes identified by the normal vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$, and line (d) gives the stress vector on a plane making an angle of 45^o with X axis. We have illustrated these points in Figs. (b) and (c). We have shown the stress vectors with thick arrows, and labelled them with the bold Greek letter $\boldsymbol{\mathcal{T}}$. We draw the following conclusion.

Conclusion:

(a) If the field is *perpendicular* to the plane, the stress vector is *normal* and *outward* (tensile stress), and equal to $\frac{\epsilon_0}{2}E^2$.

(b) If the field is *tangential* to the plane, the stress vector is *normal* and *inward* (compressive stress), and equal to $\frac{\epsilon_0}{2}E^2$.

(c) If the field makes angle 45^o to the plane, the stress vector is *tangential* (shear stress), and equal to $\frac{\epsilon_0}{2}E^2$.

Case (a) applies to a conductor in an electric field **E**. The field is perpendicular to the surface. The surface force density is the same as the stress vector. We get back the same answer as in Eq. (1) using the stress tensor, without labouring to find out what is the "external field".

4.3 Example: Force transmitted between two charged particles across a spherical boundary

We shall first obtain an expression for the **E** field at any arbitrary point P (r, θ, ϕ) located on the spherical surface Σ. The point P is at the displacement vector $\boldsymbol{\eta}$ from A and **r** from O. In order to avoid repeated appearance of the constant $\frac{1}{4\pi\epsilon_0}$, we shall set $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\boldsymbol{\mathcal{E}}$. Note that

$$\begin{aligned} \boldsymbol{\eta} &= \mathbf{r} - \boldsymbol{a} = \mathbf{r} - a\mathbf{e}_z, && (a) \\ \text{so that } \eta^2 &= r^2 + a^2 - 2ra\cos\theta, && (b) \\ \text{and } \mathbf{e}_z &= \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta. && (c) \end{aligned} \quad (85)$$

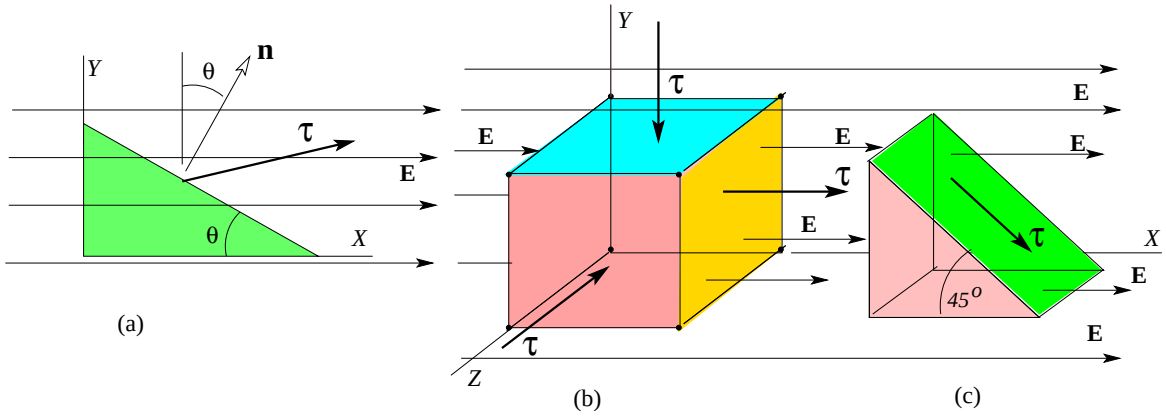

Figure 7: *Stress vector on an inclined plane placed in a uniform electric field.*

Then

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= \tfrac{Q\mathbf{r}}{r^3} + \tfrac{q\boldsymbol{\eta}}{\eta^3}. & (a)\\ &= \tfrac{Q\mathbf{e}_r}{r^2} + \tfrac{q(\mathbf{r}-a\mathbf{e}_z)}{(r^2+a^2-2ra\,\cos\theta)^{3/2}}. & (b) \end{aligned} \tag{86}$$

Therefore,

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= \mathcal{E}_r\mathbf{e}_r + \mathcal{E}_\theta\mathbf{e}_\theta, & (a)\\ \text{where}\quad \mathcal{E}_r &= \tfrac{Q}{r^2} + \tfrac{q(r-a\cos\theta)}{(r^2+a^2-2ra\,\cos\theta)^{3/2}}. & (b)\\ \mathcal{E}_\theta &= \tfrac{qa\sin\theta}{(r^2+a^2-2ra\,\cos\theta)^{3/2}}. & (c) \end{aligned} \tag{87}$$

From Eq. (79) the stress tensor is

$$\begin{aligned} \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} &= \epsilon_0\left(\mathbf{EE} - \frac{1}{2}E^2\widehat{\mathbf{1}}\right)\\ &= \frac{1}{16\pi^2\epsilon_0}\left(\boldsymbol{\mathcal{E}\mathcal{E}} - \frac{1}{2}\mathcal{E}^2\widehat{\mathbf{1}}\right) = \frac{1}{16\pi^2\epsilon_0}\tilde{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}\\ &\text{where}\quad \tilde{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} = \boldsymbol{\mathcal{E}\mathcal{E}} - \frac{1}{2}\mathcal{E}^2\widehat{\mathbf{1}}, \end{aligned} \tag{88}$$

which we may refer to as the "reduced stress tensor".

Since we have invoked the spherical coordinate system to write the expression for the $\boldsymbol{\mathcal{E}}$ field, the components of the tensor $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}$ will have to be written in this coordinate system. Since only r and θ components of $\boldsymbol{\mathcal{E}}$ are non-zero, the non-zero components of this tensor are $\mathcal{T}_{rr}, \mathcal{T}_{r\theta}, \mathcal{T}_{\theta r}, \mathcal{T}_{\theta\theta}$, as seen from (88). Therefore $\mathcal{E}^2 = \mathcal{E}_r^2 + \mathcal{E}_\theta^2$, and we write this tensor as

$$\begin{aligned} \tilde{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} &= \begin{pmatrix} \mathcal{T}_{rr} & \mathcal{T}_{r\theta} & 0\\ \mathcal{T}_{\theta r} & \mathcal{T}_{\theta\theta} & 0\\ 0 & 0 & 0 \end{pmatrix}, \text{ where}\\ \mathcal{T}_{rr} &= \mathcal{E}_r^2 - \tfrac{1}{2}\mathcal{E}^2 = \tfrac{1}{2}(\mathcal{E}_r^2 - \mathcal{E}_\theta^2).\\ \mathcal{T}_{r\theta} &= \mathcal{T}_{\theta r} = \mathcal{E}_r\mathcal{E}_\theta.\\ \mathcal{T}_{\theta\theta} &= \mathcal{E}_\theta^2 - \tfrac{1}{2}\mathcal{E}^2 = \tfrac{1}{2}(\mathcal{E}_\theta^2 - \mathcal{E}_r^2). \end{aligned} \tag{89}$$

The first column in the square matrix on the left represents the stress vector $\boldsymbol{\mathcal{T}}_r$ on the spherical surface Σ (corresponding to $\mathbf{n} = \mathbf{e}_r$, analogous to the first column in Eq. 50). Using the expressions for $\mathcal{E}_r, \mathcal{E}_\theta$ given in (87) we shall work out the components of $\boldsymbol{\mathcal{T}}_r$ explic-

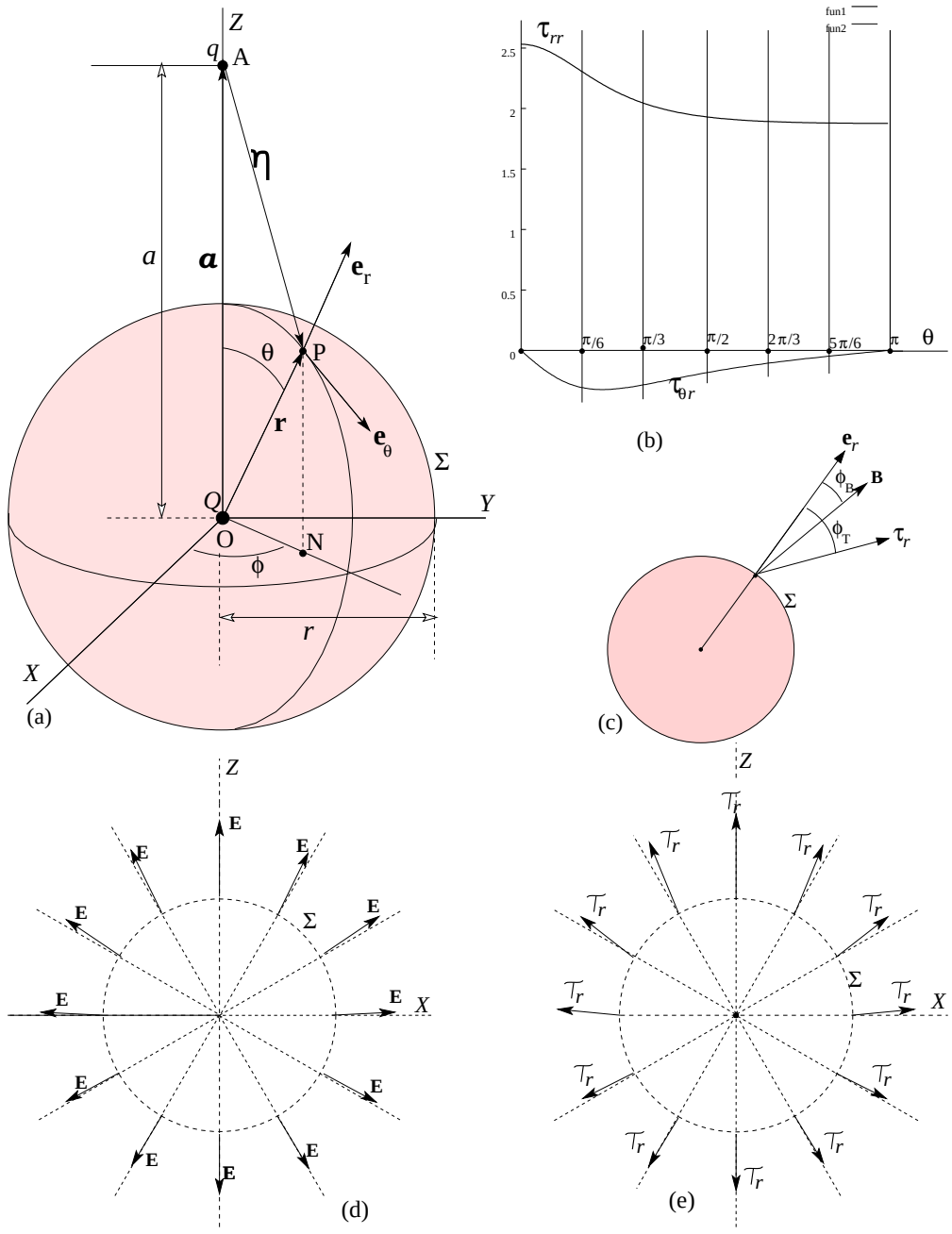

Figure 8: *Stress on a spherical surface.*

itly as follows

$$\begin{aligned}
&\boldsymbol{\mathcal{T}}_r = \mathbf{e}_r\mathcal{T}_{rr} + \mathbf{e}_\theta\mathcal{T}_{\theta r}\\
&\mathcal{T}_{rr} = \tfrac{1}{2}(\mathcal{E}_r^2 - \mathcal{E}_\theta^2) = \tfrac{1}{2}\left[\tfrac{Q^2}{r^4} + \right.\\
&\left. + \tfrac{q^2[(r-a\cos\theta)^2-(a\sin\theta)^2]}{(r^2+a^2-2ra\,\cos\theta)^3} + \tfrac{2Qq(r-\cos\theta)}{r^2(r^2+a^2-2ra\,\cos\theta)^{3/2}}\right].\\
&\mathcal{T}_{\theta r} = \mathcal{E}_r\mathcal{E}_\theta = \tfrac{Qqa\sin\theta}{r^2(r^2+a^2-2ra\,\cos\theta)^{3/2}}\\
&+ \tfrac{q^2a\sin\theta(r-a\cos\theta)}{(r^2+a^2-2ra\,\cos\theta)^3}.
\end{aligned} \tag{90}$$

The first component $\mathcal{T}_{rr}$ is the normal stress on the surface Σ and the second one $\mathcal{T}_{\theta r}$ the tangential (or, the shear) stress.

In order to illustrate the above equations, and to see how the electric field vector $\mathbf{E}$ and the Maxwell's stress vector $\boldsymbol{\mathcal{E}}$ vary on the surface of the imaginary sphere Σ, we shall make a numerical example, setting $Q = 2, q = -1, a = 3, r = 1$ in Eqs. (87) and (90). The expressions we now get are functions of the polar angle θ only. We have plotted $\mathcal{T}_{rr}, \mathcal{T}_{\theta r}$ in Fig. 8(b), using Maxima.

In order to show how the field vector $\boldsymbol{\mathcal{E}}$ and the stress vector $\boldsymbol{\mathcal{T}}_r$ vary on the surface of the sphere Σ we have prepared the Table 3.1 after evaluating the corresponding quantities in the columns 1-9, using Maxima. The angles $\phi_{\rm E}, \phi_{\rm T}$ appearing in columns 5 and 9 have been explained in Fig. 8(c). The first one is the angle between the normal $\mathbf{e}_r$ to the surface Σ and the electric field $\boldsymbol{\mathcal{E}}$ at the surface, and the second one is the angle between $\mathbf{e}_r$ and the stress vector $\boldsymbol{\mathcal{T}}_r$ on the surface.

$$\begin{aligned}
\mathcal{E} = \sqrt{\mathcal{E}_r^2 + \mathcal{E}_\theta^2}; \quad \tan\phi_{\rm E} = \frac{\mathcal{E}_\theta}{\mathcal{E}_r};\\
\mathcal{T}_r = \sqrt{\mathcal{T}_{rr}^2 + \mathcal{T}_{\theta r}^2}; \quad \tan\phi_{\rm T} = \frac{\mathcal{T}_{\theta r}}{\mathcal{T}_{rr}}.
\end{aligned} \tag{91}$$

We have drawn the field vectors $\mathbf{E}$ and the stress vectors $\boldsymbol{\mathcal{T}}_r$ on the sphere Σ in Fig. 8(d) and (e) (using two diffferent scales for the two sets of vectors.)

TABLE 3.1: $\boldsymbol{\mathcal{E}}$ AND $\boldsymbol{\mathcal{T}}_r$ VECTORS ON THE SURFACE OF THE SPHERE

1	2	3	4	5	6	7	8	9
θ	$\mathcal{E}_r$	$\mathcal{E}_\theta$	$\mathcal{E}$	$\phi_{\rm E}$	$\mathcal{T}_{rr}$	$\mathcal{T}_{\theta r}$	$\mathcal{T}_r$	$\phi_{\rm T}$
0^o	2.25	0	2.25	0^o	2.53	0	2.53	0^o
30^o	2.15	-0.14	2.16	-3.8^o	2.30	-0.31	2.33	-7.6^o
60^o	2.03	-0.14	2.03	-4^o	2.04	-0.28	2.06	-7.9^o
90^o	1.97	-0.10	1.97	-2.8^o	1.93	-0.19	1.94	-5.5^o
120^o	1.95	-0.05	1.95	-1.6^o	1.89	-0.11	1.90	-3.3^o
150^o	1.94	-0.02	1.94	-0.8^o	1.88	-0.05	1.88	-1.5^o
180^o	1.94	0	1.94	0^o	1.88	0	1.88	0^o

(92)

All this tediuos work will have been fruitful if we could show that the surface force density, when integrated over the entire surface Σ, will give us back the familiar Coulomb force between the two charges. The surface force density is the same as the stress vector on this surface. We shall work with the "reduced" surface force density, same as $\boldsymbol{\mathcal{T}}_r$.

The Coulomb force of attraction (if Q, q are of opposite signs) or repulsion (if they are of the same sign) will be along the line OA joining the two charges. Since this line coincides with the Z axis, we shall integrate the Z compomnent of $\boldsymbol{\mathcal{T}}_r$, which we shall denote as $\tilde{f}_z$. We go back to Eqs. (98) and (90) to compute this force, and get the following results after some simplification.

$$\begin{aligned}
&\tilde{f}_z = \mathbf{e}_z \cdot \boldsymbol{\mathcal{T}}_r \\
&= (\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta)\cdot(\mathbf{e}_r\mathcal{T}_{rr} + \mathbf{e}_\theta\mathcal{T}_{\theta r}) && (a)\\
&= \cos\theta\,\mathcal{T}_{rr} - \sin\theta\,\mathcal{T}_{\theta r} && (b)\\
&= \tilde{f}_z(Q^2) + \tilde{f}_z(Qq) + \tilde{f}_z(q^2), \text{ where} && (c)\\
&\tilde{f}_z(Q^2) = \tfrac{1}{2}\tfrac{Q^2}{r^4}\cos\theta. && (d)\\
&\tilde{f}_z(Qq) = \tfrac{Qq[r\,\cos\theta - a]}{r^2(r^2+a^2-2ra\,\cos\theta)^{3/2}}. && (e)\\
&\tilde{f}_z(q^2) = \tfrac{1}{2}\tfrac{q^2[(r^2+a^2)\cos\theta - 2ra]}{(r^2+a^2-2ra\,\cos\theta)^3}. && (f)
\end{aligned} \tag{93}$$

The expressions in lines (d) and (f), involving Q^2 and q^2, are "self terms", whereas the expression in line (e) involving Qq is the "interaction term" The reader should complete the steps leading from line (b) to these lines. We shall soon show that the self terms will vanish upon integration, leaving the integrated stress force entirely a function of Qq.

The "reduced" force transmitted across the surface Σ, and hence acting on the charge Q, is the surface integral of $\tilde{f}_z$. Let us denote this integral as $\tilde{F}$. An area element on Σ is $da = r^2\,\sin\theta\,d\theta\,d\phi$. Therefore,

$$\begin{aligned}
&\tilde{F} = \iint_\Sigma \tilde{f}_z\, r^2 \sin\theta\, d\theta\, d\phi \\
&= 2\pi r^2 \textstyle\int_0^\pi \tilde{f}_z\,\sin\theta\,d\theta && (a)\\
&= 2\pi r^2[\mathcal{I}(Q^2) + \mathcal{I}(Qq) + \mathcal{I}(q^2)], && (b)\\
&\text{where } \mathcal{I}(Q^2) = \textstyle\int_0^\pi \tilde{f}_z(Q^2)\,\sin\theta\,d\theta = 0. && (c)\\
&\mathcal{I}(Qq) = \textstyle\int_0^\pi \tilde{f}_z(Qq)\,\sin\theta\,d\theta = -\frac{2Qq}{a^2r^2}. && (d)\\
&\mathcal{I}(q^2) = \textstyle\int_0^\pi \tilde{f}_z(q^2)\,\sin\theta\,d\theta = 0. && (e)\\
&\text{Hence,}\quad \tilde{F} = -\tfrac{4\pi Qq}{a^2}. && (f)
\end{aligned} \tag{94}$$

The integral given in line (c) is easy to evaluate. The other integrals have been worked out in the Appendix. They can be worked out more easily using Maxima with a computer.

To get the true force we go back to (88), multiply $\tilde{F}$ with the factor $\frac{1}{16\pi^2\epsilon_0}$, and get the force $\mathbf{F}_Q$ acting on the charge Q.

$$\mathbf{F}_Q = \frac{1}{16\pi^2\epsilon_0}\tilde{F}\mathbf{e}_z = -\frac{Qq}{4\pi\epsilon_0 a^2}\mathbf{e}_z. \tag{95}$$

This force is the familiar Coulomb force on the charge Q located at the origin, exerted on it by another chareg q located at a distance a on the positive Z axis. It is repulsive, i.e., towards the negative Z axis, if Qq is positive, and attractive i.e., towards the positive Z axis, if Qq is negative.

5 Maxwell's Stress tensor for the Magnetostatic Field

This section is the magnetostatic analogue of the electrostatic stress tensor presented in Sec. 4.3. The steps are parallel, so that we shall avoid detailed explanation.

5.1 Volume force density in terms of the field

We shall construct Maxwell's stress tensor for the magnetostatic field, represent it by the symbol $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})}$. The volume force density in a magnetic field is $\mathbf{f}^{(\mathrm{M})} = \mathbf{J} \times \mathbf{B}$. Therefore we need to construct the tensor $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})}$ under the specification

$$\boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})} \equiv \mathbf{f}^{(\mathrm{M})} = \mathbf{J} \times \mathbf{B}. \qquad (96)$$

This is now an easy task, thanks to the identity (76) we had established in Sec. 4. We set $\mathbf{B}$ for $\mathbf{A}$ in that equation, and use the field equations: $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$; $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$, leading to:

$$\begin{array}{rrcll} & \mathbf{f}^{(\mathrm{M})} & = & \mathbf{J} \times \mathbf{B} = \boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})}, & (a) \\ \text{where} & \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})} & = & \frac{1}{\mu_0}\left[\mathbf{BB} - \frac{1}{2}B^2\widehat{\mathbf{1}}\right]. & (b) \end{array} \qquad (97)$$

Note the similarity between the stress tensor $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})}$ written above and the stress tensor $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})}$ written in Eq. (79) on page 24. The former converts into the latter if we replace $\mathbf{E}$ with $\mathbf{B}$ and ϵ_0 with $\frac{1}{\mu_0}$. In the same way the matrix form given in Eq. (80) converts to the matrix form of $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})}$. Consequently, the stress vector changes from normal outward, to tangential, to normal inward, as the angle between the plane and the direction of the $\mathbf{B}$ field changes from 90^o to 45^o to 0^o, as shown in Eqs. (84) and illustrated in Fig 7. and the "Conclusion" written on page 25 carries over to the case of a magnetic field without any change. Each point in the conclusion is well illustrated in Fig. 9 (see next section) if the reader compares the direction of the field vector $\mathbf{B}$ in Fig(d) with the direction of stress vector $\boldsymbol{\mathcal{T}}_r$ in Fig(e).

5.2 Example: Force transmitted between two magnetic diploles across a spherical boundary

The smallest denomination of the source of a magnetic field is a magnetic dipole, consisting of a tiny current loop. We shall therefore think of the force between two magnetic dipoles. We have placed these dipoles along the Z axis, oriented them in the positive direction of this axis. Fig. 9(a) shows the geometry of this configuration. The dipoles are shown by tiny spherical blobs with an arrow pointing in the direction of this vector. As in the electrostatic example, we shall illustrate Maxwell's stress tensor $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})}$ by finding the stress vector on the surface of an imaginary sphere Σ of radius r surrounding the point magnetic dipole $\mathbf{M}$ which is placed at a dis-

tance a from the other point magnetic dipole $\mathbf{m}$ such that $r < a$, and then integrate this stress vector over the spherical surface to obtain the force $\mathbf{F}_{\rm M}$ on $\mathbf{M}$ exerted by $\mathbf{m}$.

We shall first obtain the $\mathbf{B}$ field at any arbitrary point P (r, θ, ϕ) located on the spherical surface Σ, at the displacement vector $\boldsymbol{\eta}$ from A and $\mathbf{r}$ fromO. In order to avoid repeated appearance of the constant $\frac{\mu_0}{4\pi}$, we shall set $\mathbf{B} = \frac{\mu_0}{4\pi}\boldsymbol{\mathcal{B}}$. Note that

$$\begin{aligned} \boldsymbol{\eta} &= \mathbf{r} - \boldsymbol{a} = \mathbf{r} - a\mathbf{e}_z, & (a)\\ \text{so that } \eta^2 &= r^2 + a^2 - 2ra\,\cos\theta, & (b)\\ \text{and } \mathbf{e}_z &= \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta. & (c) \end{aligned} \tag{98}$$

Let $\boldsymbol{\mathcal{B}}^{(M)}(r,\theta,\phi)$, $\boldsymbol{\mathcal{B}}^{(m)}(r,\theta,\phi)$ be the fields[15] produced by the dipoles $\mathbf{M}$ and $\mathbf{m}$ respectively, at any coordinate point (r, θ, ϕ). Adding them we get the total field $\boldsymbol{\mathcal{B}}(r,\theta,\phi)$.

$$\begin{aligned} &\boldsymbol{\mathcal{B}}(r,\theta,\phi) = \boldsymbol{\mathcal{B}}^{(M)}(r,\theta,\phi) + \boldsymbol{\mathcal{B}}^{(m)}(r,\theta,\phi).\\ &\boldsymbol{\mathcal{B}}^{(M)}(r,\theta,\phi) = \tfrac{3\,(\mathbf{M}\cdot\mathbf{r})\,\mathbf{r} - \mathbf{M}r^2}{r^5}\\ &= \mathcal{B}_r^{(M)}\mathbf{e}_r + \mathcal{B}_\theta^{(M)}\mathbf{e}_\theta, \text{ where,}\\ &\mathcal{B}_r^{(M)} = \tfrac{2M\,\cos\theta}{r^3}; \quad \mathcal{B}_\theta^{(M)} = \tfrac{M\,\sin\theta}{r^3}.\\ &\boldsymbol{\mathcal{B}}^{(m)}(r,\theta,\phi) = \tfrac{3\,(\mathbf{m}\cdot\boldsymbol{\eta})\,\boldsymbol{\eta} - \mathbf{m}\eta^2}{\eta^5}\\ &= \mathcal{B}_r^{(m)}\mathbf{e}_r + \mathcal{B}_\theta^{(m)}\mathbf{e}_\theta, \text{ where,}\\ &\mathcal{B}_r^{(m)} = \tfrac{m[2(r^2+a^2)\cos\theta - (3+\cos^2\theta)ar]}{\eta^5},\\ &\mathcal{B}_\theta^{(m)} = \tfrac{m(r^2-2a^2+ar\cos\theta)\sin\theta}{\eta^5}. \end{aligned} \tag{99}$$

For future convenience we write

$$\begin{aligned} &\boldsymbol{\mathcal{B}} = \mathcal{B}_r\mathbf{e}_r + \mathcal{B}_\theta\mathbf{e}_\theta, \quad \text{where,}\\ &\mathcal{B}_r = \left(\tfrac{M}{r^3}\right)\alpha + \left(\tfrac{m}{\eta^5}\right)\beta. \quad \alpha = 2\cos\theta.\\ &\mathcal{B}_\theta = \left(\tfrac{M}{r^3}\right)\gamma + \left(\tfrac{m}{\eta^5}\right)\delta. \quad \gamma = \sin\theta.\\ &\beta = 2(r^2+a^2)\cos\theta - (3+\cos^2\theta)ar.\\ &\delta = (r^2 - 2a^2 + ar\cos\theta)\sin\theta. \end{aligned} \tag{100}$$

From Eq. (97) the stress tensor is

$$\begin{aligned} \widehat{\boldsymbol{\mathcal{T}}}^{(\rm M)} &= \frac{1}{\mu_0}\left(\mathbf{BB} - \frac{1}{2}E^2\widehat{\mathbf{1}}\right)\\ &= \frac{\mu_0}{16\pi^2}\left(\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{B}} - \frac{1}{2}\mathcal{B}^2\widehat{\mathbf{1}}\right) = \frac{\mu_0}{16\pi^2}\tilde{\boldsymbol{\mathcal{T}}}^{(\rm M)}\\ \text{where} \qquad \tilde{\boldsymbol{\mathcal{T}}}^{(\rm M)} &= \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{B}} - \frac{1}{2}\mathcal{B}^2\widehat{\mathbf{1}}. \end{aligned} \tag{101}$$

which we may refer to as the "reduced stress tensor". The non-zero components of this tensor needed by us are

$$\begin{aligned} \mathcal{T}_{rr} = \mathcal{B}_r^2 - \frac{1}{2}\mathcal{B}^2 = \frac{1}{2}(\mathcal{B}_r^2 - \mathcal{B}_\theta^2)\\ \mathcal{T}_{r\theta} = \mathcal{T}_{\theta r} = \mathcal{B}_r\mathcal{B}_\theta. \end{aligned} \tag{102}$$

In order to illustrate the above equations, and to see how the magnetic field vector $\mathbf{B}$ and the Maxwell's stress vector $\widehat{\boldsymbol{\mathcal{T}}}^{(\rm M)}$ look like on the surface of the imaginary sphere surrounding the charge Q, we shall make a numerical example, setting $M = 2, m = 1, a = 3, r = 1$ in Eqs. (100) and (102). For this purpose we have prepared the following table, after evaluating the corresponding quantities in the columns 1-9 using Maxima. The angles $\phi_{\rm B}, \phi_{\rm T}$ appearing in this table have been explained in Fig. 9(c). See also Eq. (90).

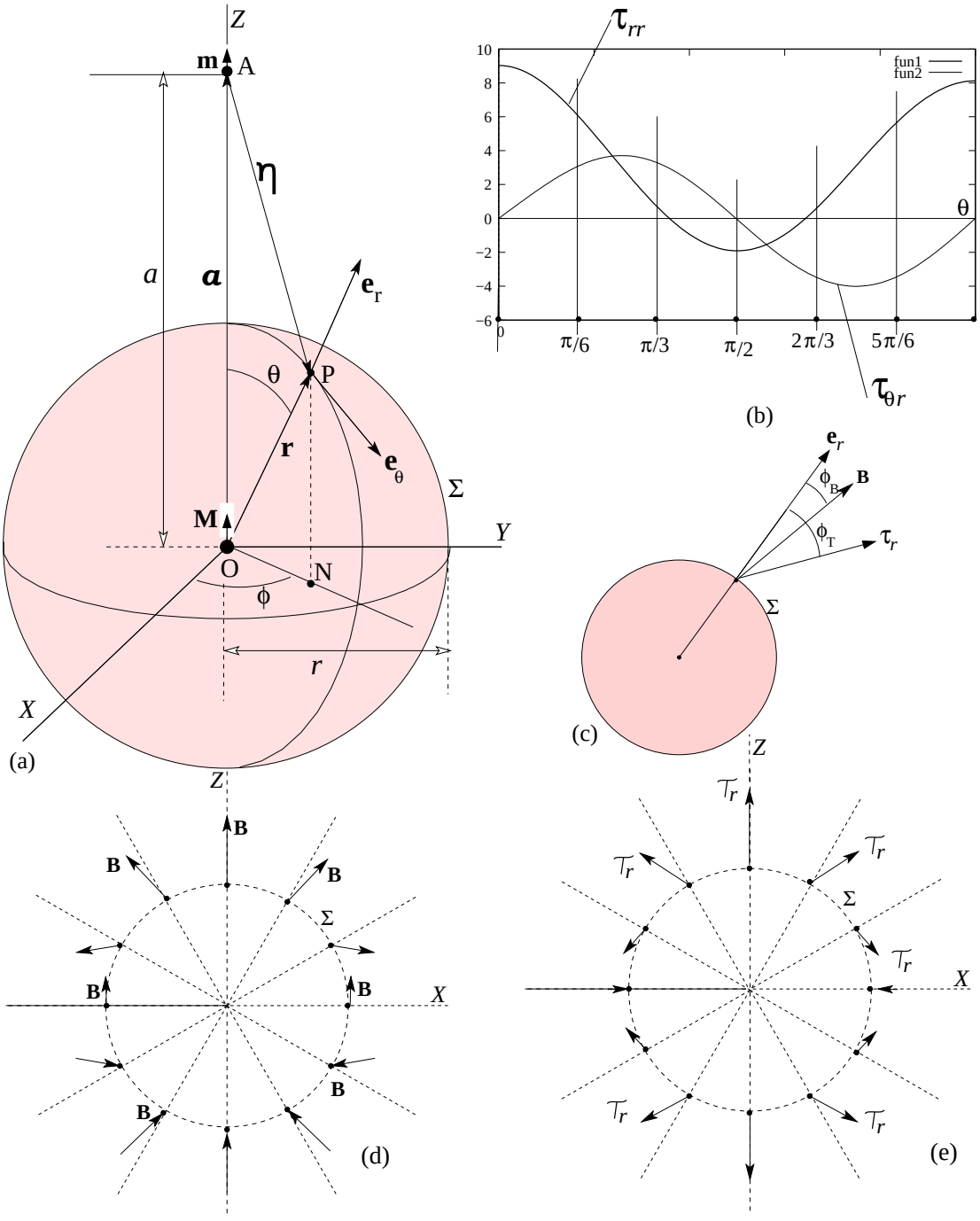

Figure 9: *Stress vector on a spherical surface.*

TABLE : $\boldsymbol{\mathcal{B}}$ AND $\boldsymbol{\mathcal{T}}_r$ VECTORS ON THE SURFACE OF THE SPHERE

1	2	3	4	5	6	7	8	9
θ	$\mathcal{B}_r$	$\mathcal{B}_\theta$	$\mathcal{B}$	$\phi_{\rm B}$	$\mathcal{T}_{rr}$	$\mathcal{T}_{\theta r}$	$\mathcal{T}$	$\phi_{\rm T}$
0^o	4.25	0	4.25	0^o	9.03	0	9.03	0^o
30^o	3.58	0.86	3.69	13.5^o	6.05	3.09	6.80	26.9^o
60^o	2.00	1.64	2.59	39.3^o	0.66	3.28	3.34	78.5^o
90^o	-0.03	1.96	1.96	-89.4^o	-1.91	-0.06	1.91	1.7^o
120^o	-2.03	1.71	2.66	-40.1^o	0.60	-3.48	3.53	-76.8^o
150^o	-3.5	0.99	3.63	-16.0^o	5.62	-3.47	6.60	-31.5^o
180^o	-4.03	0	4.03	0^o	8.13	0	8.13	0^o

(103)

We have plotted $\mathcal{T}_{rr}, \mathcal{T}_{\theta r}$ as functions of the polar angle θ in Fig. 9(b), using Maxima, and have drawn the vectors $\mathbf{B}$ and $\boldsymbol{\mathcal{T}}_r$ on the sphere Σ in Fig. 9(d) and (e) (using two diff-ferent scales for the two sets of vectors.)

All this tediuos work will have been fruitful if we could show that the surface force density, when integrated over the entire surface Σ, will yield the same force between the two dipoles that we can calculate using the standard formulas of megnetostatics. Let us then first apply the "standard formula"

The force $\mathbf{F}_{\rm M}$ on $\mathbf{m}$ is given by the formula $\mathbf{F} = (\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{B}$, in which $\mathbf{B}$ is the field created by $\mathbf{M}$. The $\mathbf{m}$ vector is in the Z direction. Therefore, $\mathbf{m}\cdot\boldsymbol{\nabla} = m\frac{\partial}{\partial z}$, which means that we can treat the (x, y) coordinates as constant and equal to zero. Therefore,

$$\begin{array}{l} \mathbf{F}_{\rm m} = m\frac{\partial \mathbf{B}}{\partial z}|_{x=y=0,z=a}, \\ \text{where, } \mathbf{B}(0,0,z) = \frac{\mu_0 M}{4\pi}\left[\frac{3z^2-z^2}{z^5}\right]\mathbf{e}_z. \\ \frac{\partial \mathbf{B}}{\partial z}|_{x=y=0,z=a} = -\frac{3\mu_0 M}{2\pi}\frac{1}{a^4}\mathbf{k}. \\ \text{Hence, } \mathbf{F}_{\rm m} = -\frac{3\mu_0 mM}{2\pi a^4}\mathbf{e}_z. \end{array} \quad (104)$$

By Newton's third law of motion,

$$\mathbf{F}_{\rm M} = -\mathbf{F}_{\rm m} = \frac{3\mu_0 Mm}{2\pi a^4}\mathbf{e}_z. \quad (105)$$

Now we shall calculate the same force using the stress tensor. The surface force density is the same as the stress vector on this surface. We shall work with the "reduced" surface force density, same as $\boldsymbol{\mathcal{T}}_r$.

The force of attraction between the dipoles will be along the line OA joining them, which lies on the Z axis. Therefore we need the Z component of the surface force density $\tilde{f}_z$:

$$\begin{aligned} \tilde{f}_z &= \mathbf{e}_z \cdot \boldsymbol{\mathcal{T}}_r \\ &= (\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta)\cdot(\mathbf{e}_r\mathcal{T}_{rr} + \mathbf{e}_\theta\mathcal{T}_{\theta r}) \\ &= \cos\theta\,\mathcal{T}_{rr} - \sin\theta\,\mathcal{T}_{\theta r}. \end{aligned} \quad (106)$$

We shall break up this force density into three components: (1) $\tilde{f}_z(M^2)$ representing self term for $\mathbf{M}$, (2) $\tilde{f}_z(Mm)$ representing interaction term between $\mathbf{M}$ and $\mathbf{m}$, (3) $\tilde{f}_z(m^2)$ representing self term for $\mathbf{m}$. From Eqs. (100), (102) and (106):

$$\begin{aligned}
\tilde{f}_z(M^2) &= \left[\left(\alpha^2-\gamma^2\right)\cos\theta - 2\alpha\gamma\sin\theta\right]\frac{M^2}{2r^6}.\\
\tilde{f}_z(Mm) &= [(\alpha\beta-\gamma\delta)\cos\theta\\
&\qquad -(\alpha\delta+\beta\gamma)\sin\theta]\tfrac{Mm}{r^3\eta^5}.\\
\tilde{f}_z(m^2) &= \left[\left(\beta^2-\delta^2\right)\cos\theta - 2\beta\delta\sin\theta\right]\frac{m^2}{2\eta^{10}}.
\end{aligned} \tag{107}$$

The "reduced" force $\tilde{F}$ transmitted across the surface Σ, and hence acting on the dipole $\mathbf{M}$, is the surface integral of $\tilde{f}_z$, which is the sum of the integrals of $\tilde{f}_z(M^2)$, $\tilde{f}_z(Mm)$, and $\tilde{f}_z(m^2)$. Each integral is difficult to evaluate, because $\alpha, \beta, \gamma, \delta$ are complicated functions of r, a, θ. We have evaluated these integrals using Maxima. The result is as follows.

$$\begin{aligned}
&\tilde{F} = \iint\limits_{\Sigma} \tilde{f}_z\, r^2\sin\theta\, d\theta\, d\phi = 2\pi r^2 \textstyle\int_0^\pi \tilde{f}_z\, \sin\theta\, d\theta\\
&= 2\pi r^2[\mathcal{I}(M^2)+\mathcal{I}(Mm)+\mathcal{I}(m^2)],\ \text{where}\\
&\mathcal{I}(M^2) = \textstyle\int_0^\pi \tilde{f}_z(M^2)\, \sin\theta\, d\theta = 0.\\
&\mathcal{I}(Mm) = \textstyle\int_0^\pi \tilde{f}_z(Mm)\, \sin\theta\, d\theta = \frac{12Mm}{a^4r^2}.\\
&\mathcal{I}(m^2) = \textstyle\int_0^\pi \tilde{f}_z(m^2)\, \sin\theta\, d\theta = 0.\\
&\text{Hence,}\quad \tilde{F} = \tfrac{24\pi Mm}{a^4}.
\end{aligned} \tag{108}$$

Because of the relation (101) the true force $\mathbf{F}_{\text{M}}$ acting on the dipole $\mathbf{M}$ is $\frac{\mu_0}{16\pi^2}$ times the force $\tilde{F}$. Hence

$$\mathbf{F}_{\text{M}} = \frac{3\mu_0 Mm}{2\pi a^4}\mathbf{e}_z. \tag{109}$$

We have thus verified that the stress tensor has given us the same force that we obtained in Eq. (105) using standard formulas of magnetostatics.

We have worked out three examples to bring out the meaning of Maxwell's stress tensor for electric and magnetic fields. The reader may wonder why we should go through such a tortuous road to get answers that can be easily obtained using simpler formulas of electrostatics and magnetostatics? Isn't it like demolishing a mud wall with a cannon?

Every cannon needs a mud wall to ensure its trust-worthiness before deployment in a true situation. Maxwell's stress tensor is destined to play a bigger role, in constructing the conservation equation for field momentum, and later under the watchful eye of Special Relativity, in building up the covariant expression for conservation of energy and momentum. The three examples we have worked out were intended to be an intellectual exercise to instil confidence in the mathematical expressions of $\widehat{\boldsymbol{\mathcal{T}}}^{(\text{E})}$ and $\widehat{\boldsymbol{\mathcal{T}}}^{(\text{M})}$ before crowning them for their majestic role.

6 Maxwell's Stress Tensor and Momentum Conservation

We had introduced Maxwell's stress tensor for static electric and static magnetic fields, with suitable applications, in Sections 4 and 5. These applications demonstrated that the force acting on static distributions of electric charges and currents lying within a bounded volume $\mathcal{V}$ is equal to the stress vector integrated over the surface $\mathcal{S}$ bounding this vol-

ume. The attribute "static" implied that the objects considered in our discussion. e.g., isolated charges and isolated current carrying loops, were fixed with a kind of "glue" making them immobile inspite of the electric and magnetic forces acting on them. We shall now remove that glue and see what role can now be played by the same stress tensors.

At this point we shall make a subtle distinction between force and stress. Force acts on material objects which may be discrete charged particles or a localized continuous material media, e.g., a plasma. The stress considered here acts on the field, which is a kind of etherial medium, as conceived by Maxwell and his contemporary physicists. In the absence of any glue holding them, the charges (e.g., electrons, nuclei) and currents (e.g., current loops) will be free to move and gain momentum. However, the momentum need not be confined to material objects. It can be shared by the field as well. Therefore we shall make the following conjecture.

Conjecture 1 *There exists a Maxwell's Stress Tensor* $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})}$ *for the Electromagnetic field, and it is given as*

$$\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})} \equiv \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} + \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{M})} = \epsilon_0 \left[\mathbf{EE} - \frac{1}{2} E^2 \widehat{\mathbf{1}} \right] + \frac{1}{\mu_0} \left[\mathbf{BB} - \frac{1}{2} B^2 \widehat{\mathbf{1}} \right], \tag{110}$$

such that

$$\frac{d}{dt} \left(\iiint_{\mathcal{V}} \mathbf{\Pi} \, d^3 r \right) + \frac{d}{dt} \left(\iiint_{\mathcal{V}} \mathbf{P} \, d^3 r \right) = \iint_{S} \widehat{\mathbf{T}}^{(\mathrm{EM})} \cdot \mathbf{n}(\mathbf{r}) \, da. \tag{111}$$

where $\mathbf{\Pi}$ *and* $\mathbf{P}$ *are, respectively, the field momentum density and the material momentum density, the latter being governed by Newton-Minkowski-Lorentz-force equation*

$$\frac{\partial \mathbf{P}}{\partial t} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}. \tag{112}$$

The right side of Eq. (111) gives the stress transmitted across the boundary $\mathcal{S}$. The right side of Eq. (112) gives the density of Lorentz force acting on all charged matter lying within the volume $\mathcal{V}$. We shall convert the surface integral on the right side of (111) into a volume intregral, using Gauss's theorem (see Sec. 3.3) so that each term in this equation is a volume integral, and then remove the integral sign reducing the same equation to an equality among three density

functions.

$$\begin{aligned} \frac{\partial \mathbf{\Pi}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} &= \boldsymbol{\nabla} \cdot \widehat{\mathbf{T}}^{(\mathrm{EM})}. \quad \text{Or,} \\ \frac{\partial \mathbf{\Pi}}{\partial t} + \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} &= \boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} + \boldsymbol{\nabla} \cdot \widehat{\mathbf{T}}^{(\mathrm{M})}. \end{aligned} \tag{113}$$

We shall now show that the above conjecture is right, that starting from Maxwell's equations we are able to find an expression for the field momentum density such that the momentum conservation of matter and field together falls into the scheme suggested in Eq. (113). Our task is made simple by the identity the identity (76) we had established in Sec. 4. We shall do the work in two stages: (1) set $\mathbf{E}$ for $\mathbf{A}$ in (76), and use Maxwell's equations: $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0$; $\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, (2) set $\mathbf{B}$ for $\mathbf{A}$ and use Maxwell's equations: $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$; $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$.

$$\begin{aligned} &\boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{E})} = \boldsymbol{\nabla} \cdot \epsilon_0 \left[\mathbf{EE} - \tfrac{1}{2}E^2\widehat{\mathbf{1}}\right] \\ &= \epsilon_0 \left[(\boldsymbol{\nabla} \cdot \mathbf{E})\mathbf{E} - \mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{E})\right] \\ &= \rho\mathbf{E} + \epsilon_0 \mathbf{E} \times \tfrac{\partial \mathbf{B}}{\partial t}. \\ &\boldsymbol{\nabla} \cdot \widehat{\mathbf{T}}^{(\mathrm{M})} = \boldsymbol{\nabla} \cdot \tfrac{1}{\mu_0} \left[\mathbf{BB} - \tfrac{1}{2}B^2\widehat{\mathbf{1}}\right] \\ &= \tfrac{1}{\mu_0} \left[(\boldsymbol{\nabla} \cdot \mathbf{B})\mathbf{B} - \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B})\right] \\ &= -\mathbf{B} \times (\mathbf{J} + \epsilon_0 \tfrac{\partial \mathbf{E}}{\partial t}) = \mathbf{J} \times \mathbf{B} + \epsilon_0 \tfrac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}. \\ &\boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})} = \tfrac{\partial}{\partial t} (\epsilon_0 \mathbf{E} \times \mathbf{B}) + (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}). \end{aligned} \tag{114}$$

The last equation is obtained by adding the first two, and using definition of $\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})}$ as given in (110). It confirms validity of our conjecture and identifies the field momentum density as

$$\boxed{\mathbf{\Pi} = \epsilon_0(\mathbf{E} \times \mathbf{B}).} \tag{115}$$

We shall like to recast Eq. (113a) into the general format of conservation equation

$$\frac{\partial}{\partial t}(\text{volume density}) + \boldsymbol{\nabla} \cdot (\textbf{flux density}) = 0. \tag{116}$$

In this case the momentum flux density $\widehat{\mathbf{\Phi}}$ is to be identified as

$$\widehat{\mathbf{\Phi}} = -\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})}. \tag{117}$$

Eq. (113a) now reads like a true momentum conservation equation:

$$\frac{\partial}{\partial t}(\mathbf{\Pi} + \mathbf{P}) + \boldsymbol{\nabla} \cdot \widehat{\mathbf{\Phi}} = \mathbf{0}. \tag{118}$$

It may be easier to comprehend the meaning of the above conservation equation by writing its three cartesian components. For example, the x-component of the above equation will be

$$\begin{aligned} &\tfrac{\partial P_x}{\partial t} + \tfrac{\partial \Pi_x}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{\Phi}_x = 0, \\ &\text{where } \mathbf{\Phi}_x = \widehat{\mathbf{\Phi}} \cdot \mathbf{e}_x = -\widehat{\boldsymbol{\mathcal{T}}}^{(\mathrm{EM})} \cdot \mathbf{e}_x \\ &= -\epsilon_0[\mathbf{e}_x \tfrac{1}{2}(E_x^2 - E_y^2 - E_z^2) \\ &+ \mathbf{e}_y E_y E_x + \mathbf{e}_z E_z E_x] \\ &- \tfrac{1}{\mu_0}[\mathbf{e}_x \tfrac{1}{2}(B_x^2 - B_y^2 - B_z^2) \\ &+ \mathbf{e}_y B_y B_x + \mathbf{e}_z B_z B_x]. \end{aligned} \tag{119}$$

The first two terms in the first line give the rate of increase of the x-component of total momentum (consisting of field momentum and material momentum) per unit volume, the third term gives the rate of outflux of the x- component of the field momentum per unit volume. Conservation of momentum implies that the sum of the two must be zero.

Before leaving this topic let us take a look at the field energy density $\mathcal{U}$ and the field energy flux density $\mathbf{S}$ (i.e. the Poyning's vector)

written in Eq. (**??**). It is immediately noticed that

$$\mathbf{S} = c^2\mathbf{\Pi}. \tag{120}$$

When the electromagnetic field is a radiation field, $E = cB$ and $\mathbf{E} \times c\mathbf{B} = E^2\mathbf{n}$ where $\mathbf{n}$ is the direction of the Poynting.s vector, giving the direction of the flow of radiation energy. For such radiation fields,

$$\mathcal{U} = \epsilon_0 E^2;\ \mathbf{S} = c\mathcal{U}\,\mathbf{n};\ \mathbf{\Pi} = \frac{\mathcal{U}}{c}\mathbf{n};\ \mathcal{U} = c\,\Pi. \tag{121}$$

The last equality is a reninder of the relation $E = cp$ between the energy E and the momentum p of a photon.

We are still not too clear about the true meaning of the momentum flux density $\widehat{\mathbf{\Phi}}$. To get familiarity with it let us consider a plane electromagnetic wave propagating in the x-direction, polarized in the y-direction. For such a field $\mathbf{E} = E\mathbf{e}_y$, $c\mathbf{B} = E\mathbf{e}_z$. It is a simple exercise to evaluate $\widehat{\mathbf{\Phi}}$ by setting $E_x = 0, E_y = E, E_z = 0$; $cB_x = 0, cB_y = 0, cB_z = E$ in the expression for $\mathbf{\Phi}_x$ in Eq. (119c) and similar expressions for $\mathbf{\Phi}_y, \mathbf{\Phi}_z$ and obtain

$$\begin{aligned}\widehat{\mathbf{\Phi}} = \mathbf{\Phi}_x\mathbf{e}_x + \mathbf{\Phi}_y\mathbf{e}_y + \mathbf{\Phi}_z\mathbf{e}_z = (\epsilon_0 E^2\mathbf{e}_x)\,\mathbf{e}_x \\ = c\,\Pi\,\mathbf{e}_x\mathbf{e}_x = c\,\mathbf{\Pi}\,\mathbf{e}_x = \mathbf{\Pi}\,\mathbf{c}.\end{aligned} \tag{122}$$

Here $\mathbf{c} = c\mathbf{e}_x$ represents the "velocity" of light, being the speed c multiplied with a unit vector in the direction of prpagation. If we now consider a plane perpendicular to the X-axis, so that $\mathbf{n} = \mathbf{e}_x$, then the outflux of field momentum per unit area across the plane will be $\widehat{\mathbf{\Phi}} \cdot \mathbf{n} = \widehat{\mathbf{\Phi}} \cdot \mathbf{e}_x = c\,\mathbf{\Pi}$.

Generalization of Eq. (122) is obvious. If there is a source of radiation at the origin (say, an antenna, or an accelereting charged particle), then far away from the origin, the momentum flux density tensor $\widehat{\mathbf{\Phi}}$ has the form

$$\widehat{\mathbf{\Phi}} = c\,\Pi\,\mathbf{e}_r\mathbf{e}_r = \mathbf{\Pi}\,c\,\mathbf{e}_r = \mathbf{\Pi}\,\mathbf{c}, \tag{123}$$

where $\mathbf{e}_r$ is the unit vector in the radial direction, also midentified with the direction of propagation of the electromagnetic wave. The tensor $\widehat{\mathbf{\Phi}}$ gives the measure of how much momentum is crossing a spherical surface per unit area per unit time. The momentum density is $\mathbf{\Pi} = \Pi\,\mathbf{e}_r$, and it is propagating in the radial direction with velocity $\mathbf{c} = c\,\mathbf{e}_r$.

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Appendix: Useful Integrals

We shall write derive the values of some integrals required in this book. The integrands of all the integrals will have in their denominators integer/half-integer powers of the expression $(r^2+a^2-2ra\,\cos\theta)$, the integration variable will be θ, and the range of integration $[0,\pi]$. We shall do some preliminary work by changing the variable of integration from θ to η, accompanied by the change of the limits of integration, and conversion of the numerators for the first two cases.

$$\begin{aligned}
\eta^2 &= r^2+a^2-2ra\,\cos\theta, && (a)\\
\eta\,d\eta &= ar\,\sin\theta\,d\theta. && (b)\\
a-r\cos\theta &= \tfrac{a^2-r^2+\eta^2}{2a} && (c)\\
(r^2+a^2)\cos\theta-2ra &= \tfrac{(a^2-r^2)-(a^2+r^2)\eta^2}{2ra} && (d)\\
\text{Lower limit: } \theta=0\Rightarrow\eta &= \{(a-r),\ \text{if } a>r\};\quad \{(r-a),\ \text{if } r>a\}. && (e)\\
\text{Upper limit: } \theta=\pi\Rightarrow\eta &= a+r. && (f)
\end{aligned} \tag{124}$$

6.1 Direct Evaluation

Using the above conversions hints it should not be difficult for the reader to establish the following integrals.

Integral # 1

$$\Psi_1(a,r) \equiv \int_0^\pi \left[\frac{(a - r\cos\theta)}{(r^2 + a^2 - 2ra\,\cos\theta)^{\frac{3}{2}}}\right] \sin\theta\, d\theta = \begin{cases} \dfrac{2}{a^2} & \text{if} \quad a > r; \\ 0 & \text{if} \quad a < r. \end{cases} \tag{125}$$

Integral # 2

$$\Psi_2(r,a) \equiv \int_0^\pi \left[\frac{(r^2 + a^2)\cos\theta - 2ra}{(r^2 + a^2 - 2ra\,\cos\theta)^3}\right] \sin\theta\, d\theta = 0. \tag{126}$$

6.2 Evaluation using Maxima

We have evaluated the following three integrals, using Maxima (version 5.13.0). We shall first write down the values of the integrals, and then show the commands used in Maxima to obtain these results.

Let us write

$$\begin{array}{ll} \alpha = 2\cos\theta; & \beta = 2(r^2 + a^2)\cos\theta - (3 + \cos^2\theta)ar. \\ \gamma = \sin\theta; & \delta = (r^2 - 2a^2 + ar\cos\theta)\sin\theta. \end{array} \tag{127}$$

Integral # 3

$$\Psi_3(r,a) = \int_0^\pi \left[\left(\alpha^2 - \gamma^2\right)\cos\theta - 2\alpha\gamma\sin\theta\right]\, \sin\theta\, d\theta = 0. \tag{128}$$

Integral # 4

$$\Psi_4(r,a) = \int_0^\pi \left[\frac{(\alpha\beta - \gamma\delta)\cos\theta - (\alpha\delta + \beta\gamma)\sin\theta}{(r^2 + a^2 - 2ra\,\cos\theta)^{5/2}}\right] \sin\theta\, d\theta = \begin{cases} \dfrac{12r}{a^4}; & (a > r) \\ 0; & (a < r). \end{cases} \tag{129}$$

Integral # 5

$$\Psi_5(r,a) = \int_0^{\pi} \left[\frac{(\beta^2 - \delta^2)\cos\theta - 2\beta\delta\sin\theta}{(r^2 + a^2 - 2ra\,\cos\theta)^5} \right] \sin\theta\, d\theta = 0. \tag{130}$$

6.3 Maxima Commands, Inputs and Outputs

We shall write the interactive commands and prompts between the user and the Maxima so that the reader can verify the values of the integrals #4 and # 5. Note the following:

1. Some output lines (e.g., %o5, %06 in Ex.#4) are spread over two lines in which the first line contains the "indices", e.g., "to the power 2". These indices get displaced and detached from the base when the output is copied into any text file. To avoid this anomaly we have brought them to one line using mathematical mode.

2. If the output is an expression of a definite integral, it is spread over seven lines (e.g., as in %o9 in Ex.#4), and the integral sign becomes unintelligible when copied. We have replaced these outputs and other outputs that appear too long and complicated with "...". All outputs except the final one are non-essential.

Input/Ouptut for Integral #4

(%i1) aa: 2*cos(x);
(%o1) 2 cos(x)
(%i2) bb: 2 *(r^2 + a^2) *cos(x) - ((cos(x))^2 +3)* a* r ;
(%o2) $2(r^2 + a^2)\cos(x) - ar(\cos^2(x) + 3)$
(%i3) cc: sin(x);
(%o3) sin(x)
(%i4) dd: (r^2 -2* a^2 + a* r *cos(x))*sin(x) ;
(%o4) $(ar\cos(x) + r^2 - 2a^2)\sin(x)$
(%i5) f: (aa*bb-cc*dd)* cos(x)-(aa*dd+bb*cc)* sin(x) ;
(%o5) cos(x) (2 cos(x) $(2(r^2 + a^2)\cos(x) - ar(\cos^2(x) + 3))$
- (a r cos(x) + $r^2 - 2a^2)\sin^2(x)$) - sin(x)
$((2(r^2 + a^2)\cos(x) - ar(\cos^2(x) + 3))\sin(x)$
+ 2 cos(x) (a r cos(x) + $r^2 - 2a^2$) sin(x))
(%i6) et: abs(sqrt(r^2+a^2 - 2*r*a*cos(x)));
(%o6) sqrt(- 2 a r cos(x) + $r^2 + a^2$)
(%i7) h: (f/(et^5))*sin(x) ;
(%o7) ....
(%i8) assume (a-r > 0) ;
(%o8) [a > r]
(%i9) 'integrate (h, x) ;
(%o9) ...
(%i10) changevar (%, et - y, y, x) ;
Is y positive, negative, or zero?
pos;
solve: using arc-trig functions to get a solution.
Some solutions will be lost.
(%o10) ...
(%i11) %,nouns;
Is sqrt(r + 2 a r + a) - sqrt(r - 2 a r + a) positive, negative, or zero?
pos;
Is r + a zero or nonzero?
nonzero;
(%o11) $-\left(\frac{\text{sqrt}(r^2 - 2ar + a^2)(48a^2r^7 + 36a^3r^6 + 8a^4r^5 + 4a^5r^4)}{r - a}\right.$
$\left.-\frac{\text{sqrt}(r^2 + 2ar + a^2)(48a^2r^7 - 36a^3r^6 + 8a^4r^5 - 4a^5r^4)}{r + a}\right.$
$+\text{sqrt}(r^2 + 2ar + a^2)(-48a^2r^6 + 12a^3r^5 - 4a^4r^4)$
$\left.- \text{sqrt}(r^2 - 2ar + a^2)(-48a^2r^6 - 12a^3r^5 - 4a^4r^4)\right)/(16a^6r^6)$

To simplify the last output (%o11), set

$$\text{sqrt}(r^2 - 2ar + a^2) = \begin{cases} (a - r) & \text{if, } (a > r) \\ (r - a) & \text{if, } (a < r) \end{cases} \tag{131}$$

and get $\dfrac{4 \times 48a^2r^7}{16a^6r^6} = \dfrac{12r}{a^4}$ for the first case and 0 for the second.

Input/Ouptut for Integral #5

(%i1) bb: 2*(r^2 + a^2)*cos(x) - ((cos(x))^2 +3)*a*r;
(%o1) 2 (r^2+ a^2) cos(x) - a r ($\cos^2(x)$ + 3)
(%i2) dd: (r^2 -2*a^2 + a*r*cos(x))*sin(x) ;
(%o2) (a r cos(x) + r^2- 2 a^2) sin(x)
(%i3) f: (bb^2-dd^2)*cos(x) - 2*bb*dd*sin(x);
(%o3) cos(x) ((2 (r^2+ a^2) cos(x) - a r ($\cos^2(x)$ + 3))^2
- (a r cos(x) + r^2- 2 a^2)^2$\sin^2(x)$) - 2 (a r cos(x) + r^2- 2 a^2)
(2 (r^2+ a^2) cos(x) - a r ($\cos^2(x)$ + 3)) $\sin^2(x)$
(%i4) ets: r^2+a^2 - 2*r*a*cos(x) ;
(%o4) - 2 a r cos(x) + r^2+ a^2
(%i5) h: (f/(ets^5))*sin(x) ;
(%o5) (sin(x) (cos(x) ((2 (r^2+ a^2) cos(x) - a r ($\cos^2(x)$ + 3))^2
- (a r cos(x) + r^2- 2 a^2)^2$\sin^2(x)$) - 2 (a r cos(x) + r^2- 2 a^2)
(2 (r^2+ a^2) cos(x) - a r ($\cos^2(x)$ + 3)) $\sin^2(x)$)))/(- 2 a r cos(x) + r^2+ a^2)^5
(%i6) assume (a-r > 0) ;
(%o6) [a > r]
(%i7) 'integrate (h, x, 0, %pi) ;
(%o7) ...
(%i8) changevar (%, abs(sqrt(ets)) - y, y, x) ;
Is y positive, negative, or zero?
pos;
solve: using arc-trig functions to get a solution.
Some solutions will be lost.
(%o8) ...
(%i9) %,nouns;
Is sqrt(r^2+ 2 a r + a^2) - sqrt(r^2- 2 a r + a^2) positive, negative, or zero?
pos;
Is r + a zero or nonzero?
nonzero;
(%o9) 0

Grappling Tensors using Maxwell's equations

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Abstract

Physics teachers always face great difficulties in introducing the many component nature of tensors to students. This article is intended to help the instructors with some convincing examples of more than 3 component physical quantities. We will also show that tensor nature of many physical quantities are both medium and interacting physical quantity dependent.

1 Introduction

For many physics students physical quantities having 3^0 components are scalars, 3^1 components are vectors and physical quantities having 3^2 or more than 9 components are tensors each obeying different mathematics. In class rooms we could suggest good and convincing examples for scalars and vectors but examples for 9 or more component physical quantities are rarely discussed except moment of inertia or conductivity[1][2]. Many books give only explicit derivation for nine component moment of inertia[1] and some other examples like Maxwell stress tensor[2](where matrix form is not given) among non relativistic systems. In relativistic cases electromagnetic field tensor which is a second rank in 4 dimensional world, is a comfortable physical quantity to be introduced but we deal with it only in postgraduate classes. Tensor form of stress and strain[3] is always beyond the scope of

undergraduate classes. Thus the teachers are always forced to say that all anisotropic media are sources of tensorial nature of many physical properties without good and convincing examples. Here in this article we show how anisotropy lead to the tensor nature of many physical quantities by taking the example of the passage of an electromagnetic wave through an anisotropic medium. We also show that the conductivity tensor for an electron passing through a conductor under electric and magnetic field will be different from the conductivity tensor when a plane electromagnetic field is passing through a conducting medium.The present approach is novel by giving a direct technique in deriving the tensor nature of some physical quantities.

A simple physical tensor

Consider a plane wave and let us find how electric field of the plane electromagnetic wave is related to magnetic field. We usually say they are mutually perpendicular or they propagate with velocity $3 \times 10^8 m/s$ etc. Now let us check how they are related mathematically from Maxwell's equations. Faraday's law gives the relation between changing electric and magnetic field as

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

For a plane electromagnetic wave

$$\vec{E} = \vec{E}_0 e^{j(\vec{k}.\vec{r}-\omega t)}$$

and

$$\vec{B} = \vec{B}_0 e^{j(\vec{k}.\vec{r}-\omega t)}$$

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, j represents complex number, ω is the angular frequency, $\vec{k}$ is the wave vector, t is the time, $\vec{E}_0$ and $\vec{B}_0$ are the complex magnitudes of electric and magnetic fields. Then ∇ operation will give $j\,\vec{k}$ and $\frac{\partial}{\partial t}$ will give $-j\,\omega$. Substituting these operators Faraday's law becomes

$$\vec{k} \times \vec{E} = \omega \vec{B}$$

Expanding component wise we get

$$\omega\, B_x = 0 + -k_z\, E_y + k_y\, E_z$$
$$\omega\, B_y = k_z\, E_x + 0 - k_x\, E_z$$
$$\omega\, B_z = -k_y\, E_x + k_x\, E_y + 0$$

This can be represented in matrix form as

$$\omega \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

This shows that actually $\vec{B}$ is coupled to $\vec{E}$ in a plane electromagnetic wave through a symmetric tensor. If the propagation direction is one dimension say z-direction then we get

$$\omega \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 0 & -k_z & 0 \\ k_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

and if we have a plane polarized wave with electric field in x-direction we get

$$B_y = \frac{E_x}{c}$$

the well known relation between the magnitude of electric and magnetic fields.

Tensors

Let us now find the tensor form of various electromagnetic physical quantities. The basic equations where tensors come in electrodynamics are $\vec{J} = \sigma\vec{E}$, $\vec{B} = \mu\vec{H}$, $\vec{P} = \alpha\vec{E}$, $\vec{M} = \chi\vec{H}$ and $\vec{D} = \epsilon\vec{H}$, where $\vec{J}$ is the current density, σ is the conductivity $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, μ is the permeability, $\vec{H}$ is the magnetizing field, $\vec{P}$ is the polarization, α is the polarisability, $\vec{M}$ is the magnetization, χ is the susceptibility , $\vec{D}$ is the displacement vector and ϵ is the permittivity.

Maxwell's equations for plane waves

The Maxwell's equations[ME] for a material medium which is magnetic, dielectric and conducting with sources are

$$\nabla.\vec{D} = \rho$$

$$\nabla.\vec{H} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

where $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ and $\vec{D} = \epsilon_0\vec{E} + \vec{P}$ and ρ is the charge density. We will require for our purpose only last two equations which can be modified as

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$$

For plane waves the ME's will become

$$\vec{k} \times \vec{E} = \omega\vec{B}$$

$$j\vec{k} \times \vec{H} = \vec{J} - j\omega\vec{D}$$

From these two ME's we can obtain the tensor form of $\mu, \sigma, \epsilon, \chi$ and α. The technique is that if we want μ which is linked with the magnetic property, we will take dielectric and electrical quantities $\vec{P}$ and $\vec{J}$ as zero and rewrite the ME's equations component wise which will naturally yield the tensor form of μ. Similarly we can find the components of all the linking or bridging tensors by suitable elimination of some unrelated quantities. We will do them one by one. Let us first find permeability tensor.

Permeability tensor

While finding permeability tensor we are only interested in a magnetic medium. So $\vec{P} = 0$ and $\vec{J} = 0$. Then the Maxwell's equations become

$$\vec{k} \times \vec{E} = \omega\vec{B}$$

$$\vec{k} \times \vec{H} = -\omega\vec{D}$$

Here $\vec{D} = \epsilon_0\vec{E}$. Then ME's modify as

$$\vec{k} \times \vec{H} = -\omega\epsilon_0\vec{E}$$

which gives

$$\vec{E} = -\frac{\vec{k} \times \vec{H}}{\omega\epsilon_0}$$

Substituting we get

$$\vec{B} = -\frac{\vec{k} \times \vec{k} \times \vec{H}}{\omega^2 \epsilon_0}$$

Expanding component wise and taking $\mu = \mu_0\, \mu_r$ we get the permeability tensor

$$\mu_r = \frac{1}{\omega^2 \epsilon_0\, \mu_0} \begin{pmatrix} k^2 - k_x^2 & -k_x\, k_y & -k_x\, k_z \\ -k_y\, k_x & k^2 - k_y^2 & -k_y\, k_z \\ -k_z\, k_x & -k_z\, k_y & k^2 - k_z^2 \end{pmatrix}$$

Then each element of the permeability tensor can be written as

$$\mu_{rij} = \frac{1}{\omega^2\, \mu_0\, \epsilon_0}[k^2 \delta_{ij} - k_i\, k_j]$$

But for a magnetic medium the magnitude of wave vector

$$k = \frac{\omega}{c}\sqrt{\mu_r}$$

Then we get

$$\mu_r = \begin{pmatrix} \mu_{rr} - \mu_{r_x} & -\mu_{r_x}\, \mu_{r_y} & -\mu_{r_x}\, \mu_{r_z y} \\ -\mu_{r_y}\, \mu_{r_x} & \mu_{rr} - \mu_{r_y} & -\mu_{r_y}\, \mu_{r_z} \\ -\mu_{r_z}\, \mu_{r_x} & -\mu_{r_z}\, \mu_{r_y} & \mu_{rr} - \mu_{r_z} \end{pmatrix}$$

Thus the nature of permeability tensor as a nine component physical quantity is exhibited naturally. Similarly we can very easily get all other tensors related to electromagnetism.

Permittivity Tensor

For a pure dielectric medium $\vec{M} = 0$ and $\vec{J} = 0$. Then we get

$$\vec{D} = -\frac{\vec{k} \times \vec{k} \times \vec{E}}{\omega^2 \mu_0}$$

which on expansion will give

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} =$$

$$\frac{1}{\omega^2 \mu_0} \begin{pmatrix} k^2 - k_x^2 & -k_x\, k_y & -k_x\, k_z \\ -k_y\, k_x & k^2 - k_y^2 & -k_y\, k_z \\ -k_z\, k_x & -k_z\, k_y & k^2 - k_z^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Then permittivity tensor will be

$$\epsilon_r = \frac{1}{\omega^2 \epsilon_0\, \mu_0} \begin{pmatrix} k^2 - k_x^2 & -k_x\, k_y & -k_x\, k_z \\ -k_y\, k_x & k^2 - k_y^2 & -k_y\, k_z \\ -k_z\, k_x & -k_z\, k_y & k^2 - k_z^2 \end{pmatrix}$$

In general

$$\epsilon_{rij} = \frac{1}{\omega^2\, \mu_0\, \epsilon_0}[k^2 \delta_{ij} - k_i\, k_j]$$

But for a pure dielectric medium $k = \frac{\omega}{c}\sqrt{\epsilon_r}$ and we will get

$$\epsilon_r = \begin{pmatrix} \epsilon_{rr} - \epsilon_{r_x} & -\epsilon_{r_x}\, \epsilon_{r_y} & -\epsilon_{r_x}\, \epsilon_{r_z} \\ -\epsilon_{r_y}\, \epsilon_{r_x} & \epsilon_{rr} - \epsilon_{r_y} & -\epsilon_{r_y}\, \epsilon_{r_z} \\ -\epsilon_{r_z}\, \epsilon_{r_x} & -\epsilon_{r_z}\, \epsilon_{r_y} & \epsilon_{rr} - \epsilon_{r_z} \end{pmatrix}$$

Polarisability Tensor

Here $\vec{M} = 0$ and $\vec{J} = 0$. Taking $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$$\vec{k} \times \vec{H} + \omega(\epsilon_0\, \vec{E} + \vec{P}) = 0$$

and substituting for $\vec{H}$

$$\vec{P} = -\frac{\vec{k} \times \vec{k} \times \vec{E}}{\omega^2 \mu_0} - \epsilon_0 \vec{E}$$

This gives the polarisability components as

$$\alpha_{ij} = \frac{1}{\omega^2\, \mu_0}[(k^2 - \epsilon_0 \omega^2 \mu_0)\delta_{ij} - k_i\, k_j]$$

Conductivity Tensor

Taking $\vec{M} = 0$ and $\vec{P} = 0$ we get

$$\vec{J} = j\frac{\vec{k} \times \vec{k} \times \vec{E}}{\omega^2 \mu_0} + \omega\epsilon_0 \vec{E}$$

which on expansion will give

$$\sigma_{ij} = \frac{j}{\omega^2 \mu_0}[(\omega^2 \epsilon_0 \mu_0 - k^2)\delta_{ij} - k_i\, k_j]$$

Here conductivity tensor components will be complex which accounts for the damping of the plane wave inside the conductor[4]

Magnetic susceptibility Tensor

As done earlier taking $\vec{P} = 0$ and $\vec{J} = 0$ we can find susceptibility tensor which will be given by

$$\chi_{ij} = \frac{1}{\omega^2 \mu_0 \epsilon_0}[(k^2 - \omega^2 \epsilon_0 \mu_0)\delta_{ij} - k_i\, k_j]$$

Discussion

Thus we could obtain the tensor form of some electrodynamic quantities. But these tensor forms are not only medium dependent but also depend on the interacting quantities and external fields acting on the medium. For tensors like moment of inertia, stress or strain they are medium dependent. We will prove this with an example given below. Consider an electron traveling in a conductor under the action of an electric and magnetic field. When there is equilibrium electron will be having a drift velocity $\mathbf{v}$ and the force acting on the electron given by Lorentz force is

$$q\mathbf{E} + q\mathbf{v} \times \mathbf{B} = \frac{m\mathbf{v}}{\tau}$$

where q is the charge of the electron and τ is the time between two consecutive collisions. Let $\frac{ne^2\tau}{m} = C$ where C is a constant. Replacing $\mathbf{v}$ by $\frac{\mathbf{J}}{-ne}$ and substituting for $\mu = \frac{e\tau}{m}$ we get

$$\mathbf{E} = C\,(\mathbf{J} - \mu\,\mathbf{B} \times \mathbf{J})$$

Let us assume that magnetic field is applied in the z direction and putting $B_z = B$ we get

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = C \begin{pmatrix} 1 & \mu B & 0 \\ -\mu B & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix}$$

This is

$$\mathbf{E} = \rho\mathbf{J}$$

where ρ is the resistivity tensor with nine components. In the absence of magnetic field

$$\rho = C$$

the scalar conductivity. The conductivity tensor is given by the inverse of this and is

$$\sigma = \frac{1}{C} \begin{pmatrix} \frac{1}{1+\mu^2 B^2} & \frac{-\mu B}{1+\mu^2 B^2} & 0 \\ \frac{\mu B}{1+\mu^2 B^2} & \frac{1}{1+\mu^2 B^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Comparing the two expressions for conductivity we find that the tensor components are different and they dependent on both material and the interacting quantity. To conclude we wish to point out that there are books[5],[6], [7], [8],[9] to understand and study tensors but books which give good examples of tensors from physics point of few are very rare. This article in an attempt bridge the gap in that direction.

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Eigenstates of a Non-Relativistic Particle in a Two-Dimensional Square Well Potential

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Abstract

In this article we obtain exact eigenstates of a non-relativistic particle in a two-dimensional square well potential. The eigenvalue equations are solved using Mathematica. The eigenvalue spectrum is compared with those in one and three-dimensional analogues. Possible applications of the results are discussed.

1 Introduction

Exactly solvable eigenvalue problems in quantum mechanics are of great intrinsic interest and serve as the starting point for approximate solution of a variety of problems for which no exact solution can be found. From a detailed study of relatively simple one dimensional problems one can see clearly many properties of general validity and this helps in understanding of more complex systems of higher dimensions. In the realm of non-relativistic quantum theory, while quantum wells in one and three space dimensions are commonly discussed in literature[1, 2, 3, 4], discussion of quantum well in two space dimensions is not available. In certain semiconductors, it is possible to create heterostructure where the electron seems to be in a two-dimensional potential well[5, 6]. Hence it would be interesting to explore the bound states of a non-relativistic particle in a two-dimensional square well potential.

2 Two-dimensional Square Well Potential

The time-independent Schrödinger equation for a particle of mass m in a circularly symmetric potential $V(|\vec{r}|)$ is

$$\nabla^2\psi(\vec{r}) + \frac{2m}{\hbar^2}\left(E - V(|\vec{r}|)\right)\psi(\vec{r}) = 0 \quad (1)$$

This may be separated into two equations in cylindrical coordinates as

$$\frac{d^2\Phi}{d\phi^2} + \nu^2\Phi(\phi) = 0 \quad (2)$$

and

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{\nu^2}{\rho^2} + \frac{2m}{\hbar^2}\left(E - V(\rho)\right)\right]R(\rho) = 0 \quad (3)$$

Where $\psi(\vec{r}) = R(\rho)\Phi(\phi)$ and ν is the separation constant.
From the physical requirement of single-valuedness of the wavefunction, it follows that $\nu = \pm 0, \pm 1, \pm 2, ...$
The square well type potential of depth V_0 and radius a in two space dimension is defined by

$$V(\rho) = -V_0 \quad \text{for} \quad 0 \le \rho \le \text{a} \quad \text{Region II}$$
$$V(\rho) = 0 \quad \text{for} \quad \rho > \text{a} \quad \text{Region I}$$

The above potential is schematically shown in figure 1.
We shall look for bound states of such a potential which correspond to $E \le 0$.
In region I, the radial equation satisfies the equation

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{\nu^2}{\rho^2} - \alpha^2\right]R(\rho) = 0 \quad (4)$$

where we put $\alpha^2 = -\frac{2mE}{\hbar^2} \ge 0$, since bound states exist only for negative energies.
Putting $\alpha\rho = y$, equation (4) reduces to

$$y^2\frac{d^2R}{dy^2} + y\frac{dR}{dy} - \left(y^2 + \nu^2\right)R(y) = 0 \quad (5)$$

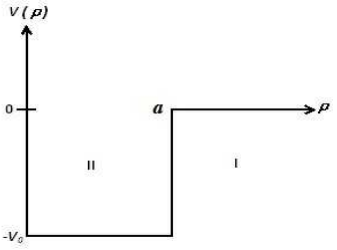

Figure 1: Schematic representation of potential well in two space dimensions

The physically admissible solution of the above equation is the one which goes to zero as $\rho \to \infty$. Hence

$$R_I(y) = C_1 K_\nu(y) \tag{6}$$

where C_1 is the normalisation constant and K_ν is the modified Bessel function of the order ν.
The other solution $I_\nu(y)$ is rejected as it is a diverging function of y.
For $\rho \leq a$, where particle is essentially confined, the radial wavefunction satisfies the second order differential equation

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{\nu^2}{\rho^2} + \beta^2\right] R(\rho) = 0 \tag{7}$$

with

$$\beta^2 = \frac{2m}{\hbar^2}(E + V_0) \geq 0 \tag{8}$$

This is clearly satisfied since $|E| < V_0$ for non trivial solutions.
Setting $\beta\rho = z$, equation (7) could be transformed into

$$z^2\frac{d^2R}{dz^2} + z\frac{dR}{dz} + \left(z^2 - \nu^2\right) R(z) = 0 \tag{9}$$

The physically acceptable solution that remains finite at $\rho = 0$ is

$$R_{II}(z) = C_2 J_\nu(z), \tag{10}$$

where $J_\nu(z)$ are the Bessel functions of the first kind. Neumann function, the other solution, being irregular at the origin is excluded.
The equation for energy eigenvalues is obtained by matching the wavefunctions in the two regions and their first derivatives at the potential boundary, i.e. $\rho = a$.
From the continuity of the wavefunctions at $\rho = a$, we obtain

$$C_1 K_\nu(\alpha a) = C_2 J_\nu(\beta a) \tag{11}$$

Requiring the derivatives of the wavefunctions also to be continuous at $\rho = a$, we obtain

$$C_1 \alpha K'_\nu(\alpha a) = C_2 \beta J'_\nu(\beta a) \tag{12}$$

Eliminating the constants in the above equations, we get

$$\alpha\frac{K'_\nu(\alpha a)}{K_\nu(\alpha a)} = \beta\frac{J'_\nu(\beta a)}{J_\nu(\beta a)} \tag{13}$$

Defining $\alpha a = \xi$ and $\beta a = \eta$, The equation may be written as

$$\xi\frac{K'_\nu(\xi)}{K_\nu(\xi)} = \eta\frac{J'_\nu(\eta)}{J_\nu(\eta)} \tag{14}$$

It is seen that $K_\nu(z)$ is related to $Y_\nu(z)$ as[7]

$$K_\nu(z) = -\frac{\pi}{2} Y_\nu(z) \tag{15}$$

$Y_\nu(z)$ are the Bessel functions of the second kind, also called Webber's function.

By virtue of above identity, equation (14) becomes

$$\xi\frac{Y_\nu'(\xi)}{Y_\nu(\xi)} = \eta\frac{J_\nu'(\eta)}{J_\nu(\eta)} \tag{16}$$

The eigenenergies corresponding to different states are obtained as solutions of the above equation.

3 Ground State Energy Spectrum

Initially, we put $\nu = 0$ and consider the $s-$state energies. We obtain, from equation (16)

$$\xi\frac{Y_0'(\xi)}{Y_0(\xi)} = \eta\frac{J_0'(\eta)}{J_0(\eta)} \tag{17}$$

Further noting that

$$J_0'(z) = -J_1(z) \text{ and } \mathrm{Y}_0'(\mathrm{z}) = -\mathrm{Y}_1(\mathrm{z}), \tag{18}$$

the eigenvalue equation may be written as

$$\xi\frac{Y_1(\xi)}{Y_0(\xi)} = \eta\frac{J_1(\eta)}{J_0(\eta)} \tag{19}$$

The solutions of this transcendental equation give the s-state eigenenergies. Numerical solutions are obtained using Mathematica [8]. We also have the equation

$$\xi^2 + \eta^2 = \frac{2mV_0a^2}{\hbar^2} \tag{20}$$

The eigenenergies may also be found by graphical method [9] using equations (19) and (20) .

Denoting

$$\frac{V_0a^2}{\frac{\hbar^2}{2m}} = r_0^2 \tag{21}$$

Equation (19) may be written as

$$\eta = \frac{\xi Y_1(\xi)\, J_0\left(\sqrt{r_0^2-\xi^2}\right)}{Y_0(\xi)\, J_1\left(\sqrt{r_0^2-\xi^2}\right)} \tag{22}$$

with

$$\xi^2 + \eta^2 = r_0^2 \tag{23}$$

The intersections of the curve for η as a function of ξ, with the circle of known radius r_0 give energy eigenvalues.

4 Energy Spectrum of Excited States

The energy eigenvalues for non-zero angular momentum states are obtained by solving equation (16) with $\nu = 1, 2, 3, 4..$ which corresponds to $p, d, f, g...$ states respectively.
We use the following identity for the derivatives of Bessel functions

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k \{z^\nu J_\nu(z)\} = z^{\nu-k}J_{\nu-k}(z) \tag{24}$$

4.1 $p-$state energy eigenvalues

For p states we put $k = 1$ and $\nu = 1$. It is straightforward to check

$$zY_1'(z) = zY_0(z) - Y_1(z) \tag{25}$$

With the above relation equation (16) reduces to

$$\frac{\xi Y_0(\xi) - Y_1(\xi)}{Y_1(\xi)} = \frac{\eta J_0(\eta) - J_1(\eta)}{J_1(\eta)} \tag{26}$$

which may be written as

$$\xi\frac{Y_0(\xi)}{Y_1(\xi)} = \eta\frac{J_0(\eta)}{J_1(\eta)} \tag{27}$$

4.2 $d-$state energy eigenvalues

Eigenenergies corresponding to $d-$states are obtained by setting $k = 1$ and $\nu = 2$.
From equation (24), we obtain

$$z\ J_2'(z) + 2J_2(z) = zJ_1(z) \tag{28}$$

which combined with equation (16) gives

$$\xi \frac{Y_1(\xi)}{Y_2(\xi)} = \eta \frac{J_1(\eta)}{J_2(\eta)} \tag{29}$$

4.3 $f-$state energy eigenvalues

Following the same procedure as before and setting $k = 1$ and $\nu = 3$, we obtain the eigenvalue equation for $f-$ state as

$$\xi \frac{Y_2(\xi)}{Y_3(\xi)} = \eta \frac{J_2(\eta)}{J_3(\eta)} \tag{30}$$

It is interesting to see that the eigenvalue equation corresponding to the excited states may well be generalised and written in a compact form as

$$\xi \frac{Y_i(\xi)}{Y_{i+1}(\xi)} = \eta \frac{J_i(\eta)}{J_{i+1}(\eta)} \tag{31}$$

where the index i runs from $0, 1, 2, 3, ...$ for $p, d, f, g, ...$ states.
Numerical energy eigenvalues for s, p, d, f, g states as a function of V_0a^2 are listed in Table 1. Both energy (E) and well-depth (V_0) are expressed in units of $\frac{\hbar^2}{2ma^2}$. Mathematica is used to extract the roots of the eigenvalue equations.
In table Table 2, we have made a comparative study of the eigenenergies of a non-relativistic particle in a finite potential well of one, two and three space dimensions. Here also, both energy (E) and well-depth (V_0) are expressed in units of $\frac{\hbar^2}{2ma^2}$. Mathematica is used to extract the roots of the eigenvalue equations. This table of eigenenergies confirms the subtle aspects of binding criteria.

5 Results and Discussions

As expected, the number of bound states increases with V_0a^2. There is a minimum value of V_0a^2 below which no bound states are possible. This is similar to the result in three space dimensions. Also, we find higher angular momentum bound states appearing at higher V_0a^2. In fact, they appear between the s states.Our results may be used to understand the bound states in two-dimensional quantum wells. Extension to an array of such potentials is under investigation.
As expected, the number of bound states increases with V_0a^2. There is a minimum value of V_0a^2 below which no bound states are possible. This is similar to the result in three space dimensions. Also, we find higher angular momentum bound states appearing at higher V_0a^2. In fact, they appear between the s states.Our results may be used to understand the bound states in two-dimensional quantum wells. Extension to an array of such potentials is under investigation.

Table 1:Energy eigenvalues of a non-relativistic particle in a 2D potential well

V_0a^2	s	p	d	f	g
1	-0.02920202				
4	-0.469381193				
9	-1.10796676				
	-7.14161486	-1.80751891			
16	-0.04526597				
	-3.11840281				
	-11.7384353	-3.94244851	-2.71043539		
25	-0.62205399				
	-7.49987519	-6.25660174			
	-18.24220979	-17.52159140	-7.21078240		
100	-1.63461339				
	-17.37414133	-4.43860624			
	-39.70411325	-20.01282854	-14.49949315		
	-61.83132962	-38.01339025	-39.28381258	-17.42820358	
	-79.90446954	-60.37600804	-62.37208576	-37.33723258	-35.69919101
	-94.79417461	-83.28460608	-80.47202318	-61.70825181	-62.52697476

Table2: Comparison of the s state energy eigenvalues of a non-relativistic particle in a finite potential well

V_0a^2	$1D$	$2D$	$3D$
1	-0.453753	-0.029202	–
4	-1.532611	-0.469381	-0.407101
9	-7.630821	-7.141614	-3.806784
	-6.532610	-1.107966	
12	-10.52852	-9.02385	-6.32137
	-9.53327	-1.66947	
	-0.62752		
16	-14.443161	-11.738434	-9.876461
	-13.532604	-3.118403	
	-3.073780	-0.045266	

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Monolayer, bilayer and trilayer graphene

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Abstract

Graphene is a single two-dimensional (2D) atomic layer of hexagonally packed carbon atoms. In field of material science all have vast interest on graphene for its numerous unique mechanical, chemical, electronic, optical and thermal properties. It has been observed that with increasing number of layers graphene show many interesting properties. In this article, we briefly discuss the structures, some properties and applications of monolayer, bilayer and trilayer graphene.

1. Introduction

Antoine Lavoisier coined the term "Carbone" first in his book "Traite Elementire de Chimie" as one of the newly identified chemical elements around 225 years back [1]. Graphene is recently discovered two-dimensional allotropic from of carbon. Before graphene three-dimensional (diamond, graphite), one-dimensional (nanotubes) and zero-dimensional (fullerenes) allotropes of carbon were known. Two carbon allotropes – diamond and graphite – have been known to humans since many centuries ago. Fullerenes were discovered in1985, carbon nanotubes in 1991 and graphene in 2004. Carbon is the only compound which has four dimension structures. i.e., 0D, 1D, 2D and 3D. The theory of graphene was first explored by P. R. Wallace [2] in 1947 as a starting point for understanding the electronic properties of more complex, 3D graphite. According to Landau and Peierls atoms in 2D crystals are displaced from its equilibrium position due to the thermal fluctuations [3,4] and this displacement is comparable with the interatomic distance at finite temperature. Moreover experimentally it is prove that the melting temperature of thin films rapidly decreases with the decreasing thickness. So in a film when there exist near about 12 layers [5] it becomes unstable, so they should not exit. Hence, the existence of graphene is a miracle! [6,7].

In 2004 [8], Andre Geim and Kostya Novoselov managed to extract single atom thick crystallite called graphene. Prof. Andre K. Geim, University of Manchester, UK and Prof. Konstantin S. Novoselov, University of Manchester, UK received the 2010 Nobel Prize in physics "for groundbreaking experiments regarding the two-dimensional material graphene". Graphene is the building block for carbon materials of all other dimensions. Graphite is obtained by the stacking of graphene layers. Diamond can be obtained from graphene under extreme pressure and temperatures by transforming the 2-dimensional sp^2 hybridization into 3-dimensional sp^3 hybridization and pi (π) bond into sigma (σ) bond. Carbon nanotubes are synthesized from rolled up graphene. Fullerenes can also be obtained from graphene by modifying the hexagons into pentagons and heptagons in a systematic way. Since its discovery graphene opens a new phase in the history of research and application in the field of material science. When graphene was discovered, it was monolayer. When several graphene layers are stacked on top of each other, the character of the charge carriers changes with the number of layers and type of stacking. Now-a-days research is going on multilayer graphene along with monolayer graphene (MLG). It has been observed that with increasing number of layers graphene show many interesting properties. For instance, monolayer graphene has zero band

gap whereas bilayer and certain types of trilayer graphene have an electrically tunable band gap [9-11]. In this article, we briefly discuss the structures, some properties and applications of monolayer, bilayer and trilayer graphene.

2. Monolayer graphene

Graphene, the hexagonally structured two-dimensional crystal, consisting of bipartite lattice of two triangular sublattices. One carbon atom of the hexagon is bonded by sigma (σ) bonds with three nearest carbon atoms and having a bond angle between them of 120°. The electronic configuration of carbon is $1s^2 2s^2 2p^2$. The three valance electrons of 2s, $2p_x$ and $2p_y$ orbital are bounded by sp^2 hybridization which results these sigma (σ) bonds. The slippery, soft nature of graphene is due to the pi (π) bond created due to overlapping of the half filled $2p_z$ orbital of one carbon atom with other $2p_z$ orbitals of other carbon atoms. This pi (π) bond also imparts electrical and thermal conductivity and luster in graphene [12-16]. In a unit cell of monolayer graphene (MLG) there are two carbon atoms which form a honeycomb structure. It has three sites and can be denoted as A, B and C (Fig. 1).

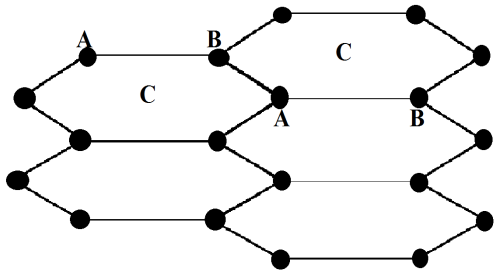

Fig. 1: Structure of monolayer graphene with A, B and C sites [25].

Electrons in monolayer graphene are obeying a linear dispersion relation (i.e. the electron energy is linearly proportional to the wave vector, $E = \hbar k v_F$) and behave as massless relativistic particles, called *Dirac fermions*. $\hbar = h/2\pi$, h is the Planck's constant, k is the wave vector and v_F is the Fermi velocity of electron in the graphene. This property implies that the speed of electrons in graphene is a constant, independent of momentum, like the speed of photons is a constant *c*. It is found that the velocity of electrons in graphene is about $10^6 m s^{-1}$. In monolayer graphene the interactions among electrons are extremely strong and graphene's dimensionless coupling constant $\alpha_{GR} = e^2/\hbar v_F \approx 1$ is larger than the dimensionless coupling constant of quantum electrodynamics (QED), $\alpha = e^2/\hbar c \approx 1/137$. Again the interaction of electrons in graphene is also different from an ordinary nonrelativistic 2D electron gas. The electrons in most of the conductors can be described by non-relativistic quantum mechanics but the electrons in graphene are treated as relativistic particles and are described by the Dirac equation rather than Schroedinger equation. This shows a possibility of studying phenomena of quantum field theory in condensed matter physics. All this makes graphene a new type of electronic system whose independent particles move relativistically, but interact nonrelativistically.

At room temperature monolayer graphene has electron mobility is 2.5×10^5 cm^2 V^{-1} s^{-1}, Young Modulus of 1 TPa; Intrinsic strength 130 GPa; High thermal conductivity of 3,000 W mK^{-1} or more [17]. It is stronger than diamond, more conductive than copper and more flexible than rubber. Monolayer graphene is used in electronics, as high frequency transistor, logic and RF transistor. Graphene can be used as non-conventional graphene switches due to its high electron mobility and high thermal conductivity.

3. Bilayer graphene

Bilayer graphene (BLG) is Bernal AB-stacked. In bilayer graphene, the Bernal stacking two layers consists of two coupled honeycomb lattices with basis atoms. The atoms are named as (A_1, B_1) and (A_2, B_2) as shown in Fig. 2. (A_1, B_1) is placed in the bottom and (A_2, B_2) in the top layers respectively.

The arranged atoms are in (A_2, B_1) fashion. That is, the A-carbon of the upper sheet lies on top of the B-carbon of the lower one [18]. The C-C bond distance is 1.42 Å and the distance between two adjacent layers is 3.35 Å.

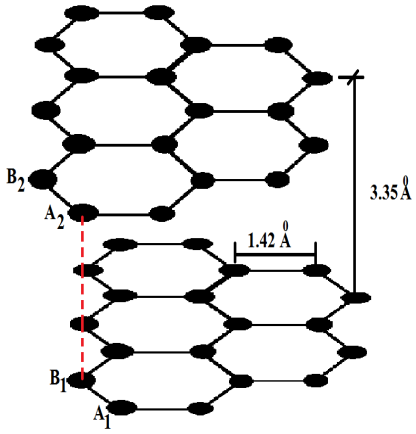

Fig. 2: Structure of bilayer graphene with Bernal AB stacking.

Electrons in MLG behave as massless two-dimensional fermions whereas electrons in BLG behave as massive two-dimensional fermions [19,20]. The density of states D as a function of particle number (or particle density) n is found to be $D(n) \sim n^{1/2}$ for massless 2d fermions (electrons in MLG) and $D(n) \sim$ constant for massive 2d fermions (electrons in BLG). Formulae for density of states are $D = \frac{1}{\hbar v_F}\sqrt{\frac{g\,n}{\pi}}$ (massless) and $D = \frac{g\,m^*}{2\pi\hbar^2}$ (massive), where g = degeneracy, and m^* is the effective mass of electrons. Thus, density of states depends upon particle density in MLG but it is independent of particle density in BLG. Massless particles (e.g. photons) have energies which depend linearly on quantum number, while the energies of massive particles (e.g. free electrons) depend quadratically on quantum number. The massless and massive dispersion relations in MLG and BLG respectively are: $E = \hbar v_F \left|\vec{k}\right|$ (massless) and $E = \frac{\hbar^2 k^2}{2m^*}$ (massive), where $\hbar = h/2\pi$, h is the Planck's constant, $\vec{k}$ is the wave vector and v_F is the Fermi velocity of massless electrons in the MLG. Monolayer graphene is more transparent than two or more layer graphene. MLG is twice transparent than BLG. Graphene shows very interesting behavior in the presence of a magnetic field at very low temperature. Graphene shows an *anomalous quantum Hall effect* (QHE) with the sequence shifted by 1/2 with respect to the standard sequence. It is found [6,14,16] that the Hall conductivity $\sigma_{xy} = \pm 4e^2/h\left(N + \frac{1}{2}\right)$, where N is the Landau level index and the factor 4 accounts for graphene's double spin and double band (valley) degeneracy. That is why; it is also characterized as *half-integer quantum Hall effect*. The QHE in bilayer graphene is more interesting. The quantized plateaus appear at the standard sequence $\sigma_{xy} = \pm 4Ne^2/h$ (same as the nonrelativistic electrons). The mobility of bilayer graphene (BLG) (~2.0 × 10^5 cm^2 V^{-1} s^{-1}2.5 × 10^5 cm^2 V^{-1} s^{-1}) is normally lower than the mobility of MLG [21–23]. Young's modulus of bilayer graphene is estimated to be 0.8 TPa which is close to the value for graphite [24]. Room-temperature thermal conductivity would be as high as about 5000 W m/K. Bilayer graphene can be used as a substitute of biological tissues, energy generation and storage, sensors and metrology.

4. Trilayer graphene

Trilayer graphene (TLG) has more complex (in comparison to monolayer and bilayer graphene) interlayer interactions that supply richer electronic structure. In trilayer graphene generally three types of stacking are possible. These are AAA, ABA and ABC stacking types corresponds to hexagonal, Bernal and rhombohedral graphene respectively (Fig. 3) [25]. Minor reflection symmetry about the centre layer among z-direction is observed in hexagonal and Bernal trilayer graphene. But rhombohedral trilayer graphene has inversion symmetry. It is observed that especially in low energy region, electronic structure of graphene is very sensitive to thickness and stacking sequence.

In case of AAA stacking all the atoms of the three layers lays one over another. For ABA stacking same atoms of the first and the third layer are exactly one over another but for the second layer that particular atom is not align. No particular type of atom in ABC stacking is in aligning with one another. The low energy band structure of ABA-stacked trilayer graphene consists of one massless and two massive subbands, similar to the spectrum of one monolayer and one bilayer graphene, whereas ABC-stacked trilayer graphene band structure is approximately cubic [26]. The AAA-stacked trilayer graphene band structure is a superposition of the band structure of its component monolayer graphenes.

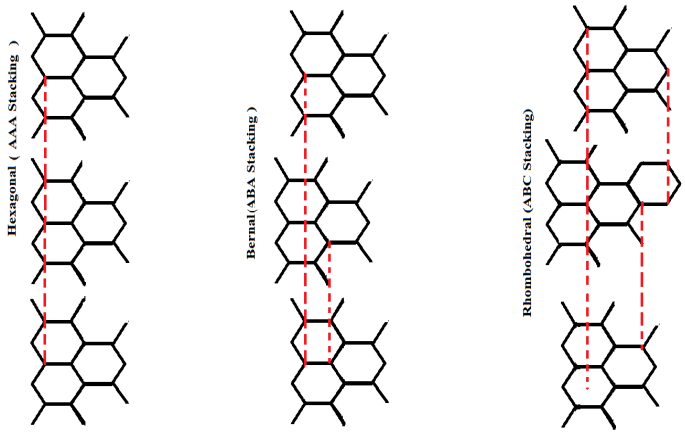

Fig. 3: Structure of trilayer graphene hexagonal (AAA), Bernal (ABA) and rhombohedral (ABC) stacking [25].

The thermodynamic properties of the electron gas in multilayer graphene depend strongly on the number of layers and the type of stacking [27,28]. At room temperature, it is observed that thermal conductivity of graphene decreases with increase in number of layers. When a strong magnetic field is applied perpendicular to the trilayer graphene planes, quantum Hall effect is observed in the material. It is found that [26] the Hall conductivity $\sigma_{xy} = \pm 4e^2/h\left(N + \frac{n}{2}\right)$, where $n = 3$ is the number of layers, N is the Landau level index, the factor 4 accounts for graphene's double spin and double valley degeneracy, $-e$ is the electron charge and h is the Planck's constant. Further, the plateau structure in σ_{xy} of trilayer graphene depends on the type of stacking. Trilayer graphene exhibits lower mobility (typically 800 cm^2 V/s at 4.2 K) compared to monolayer and bilayer graphene. Trilayer graphene can be used as LEDs, reinforcing materials, wirings, solar power aircrafts, aerospace etc.

5. Conclusions

Graphene – a two-dimensional nanomaterial, composed by covalently bonded carbon atoms in a honeycomb lattice, has been attracting the attention of the scientific community since its discovery in 2004. Due to its outstanding electronic, thermal, optical and mechanical properties it has applications in wide variety of fields, for example, spintronics, electron optics, photonics and many others. We hope, in near future it will be used in construction and telecommunication field.

Bilayer and trilayer graphene are interesting because they possess different and unique properties with respect to monolayer graphene. For example, monolayer graphene has zero band gap whereas bilayer and certain types of trilayer graphene have an electrically tunable band gap. This would make them good candidates for application in electronic industry where the control of the band gap is desirable.

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The Time evolution of a Square Wave Packet and a Triangular Wave Packet

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Abstract

In this article, we discuss the time evolution of a square wave packet and a triangular wave packet. The approach followed in this study is to express a square wave packet and a triangular wave packet as a sum of several Gaussian wave packets. Specifically, the time evolution of a square wave packet has been derived here with three and five Gaussian wave packets; then the time evolution of a triangular wave packet has been derived with three Gaussian wave packets. Their evolution with time has been plotted using MatLab™ over appropriately chosen time intervals. The results are compared with those of a Gaussian wave packet.

Keywords:- Wave packet, Schrödinger equation, Gaussian wave packet, Triangular Wave Packet, Time Evolution, MatLab™.

1. Introduction

Wave packets are superposition of plane waves used in representing a particle. According to de Broglie's matter-waves hypothesis, material particles such as photons and electrons exhibit wave nature and show wave phenomena such as interference and diffraction. For a localized particle, the superposition of many plane waves results in a function called the wave function ψ. The wave packets are decomposed by Fourier Transformation and their time evolution is found which is of physical interest. In this article, the time evolutions of non-Gaussian wave packets such as the ones mentioned in the abstract are found. The time evolutions of square wave packet and the triangular wave packet are of interest as they are often encountered in wave analyses. Using Green's function approach, Mita (2007) shows that the probability amplitude of any non-Gaussian wave packet approximately becomes a Gaussian as it disperses [1]. Here we obtain the same result using a simpler approach of approximating a square wave packet and a triangular wave packet as a sum of several Gaussian wave packets. Mita (2007) points out the following advantages of using a Gaussian wave packet:

a) A Gaussian function is easy to analyze in closed form

b) The Fourier Transform of a Gaussian is also a Gaussian and

c) The Gaussian wave packet gives rise to a minimum uncertainty product at time t=0. [1]

In order to study the time evolution, we use the following form of the Gaussian wave packet as given by Greiner, W (2004) [2]. At time t=0,

$$\psi(x,0) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2} + ik_0x\right] \quad (1)$$

where the wave function depends on the position and time coordinates, σ is the standard deviation from the mean μ, the term $\frac{1}{\sigma\sqrt{2\pi}}$ is the amplitude of the wave packet and k_0 is the wave number.

For the sake of comparison, we write the expressions or the time evolution and the probability distribution of the Gaussian wave packet. We have also plotted the time evolution of the Gaussian wave packet using MatLab™ for the sake of comparison. The evolution of the Gaussian wave packet at time t is given by

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}\sigma\left(1+\frac{i\hbar t}{m\sigma^2}\right)^{\frac{1}{2}}} exp\left(\frac{2\sigma^2 ik_0x - \frac{i\hbar t{k_0}^2\sigma^2}{m} - \frac{2\hbar tk_0\mu}{m}}{2\sigma^2\left(1+\frac{i\hbar t}{m\sigma^2}\right)} \frac{-(x-\mu)^2}{}\right) \quad (2)$$

The probability distribution is given by

$$|\psi(x,t)|^2 = \frac{\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)^{-\frac{1}{2}}}{2\pi\sigma^2} exp\left(-\frac{\left\{(x-\mu)-\frac{\hbar tk_0}{m}\right\}^2}{\sigma^2\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)}\right) \quad (3)$$

where $\hbar = \frac{h}{2\pi}$; h is the Planck's constant. This is also a Gaussian distribution with width $\sigma = \sqrt{1+\left(\frac{\hbar t}{m\sigma^2}\right)^2}$. We have assumed k_0 to be zero in order to simplify the calculation implying that the wave packet is at rest. The following values were used to plot the expression (3) in MatLab™:

Planck's constant $h = 6.6X10^{-34}$ Js

Mass of electron $m = 9.1X10^{-31}$kg

Mean value for the Gaussian wave μ = 0

Standard deviation of the wave σ = 0.5.

The following graph was obtained when the expression (3) was plotted with the above mentioned numerical values was plotted for different values of time t.

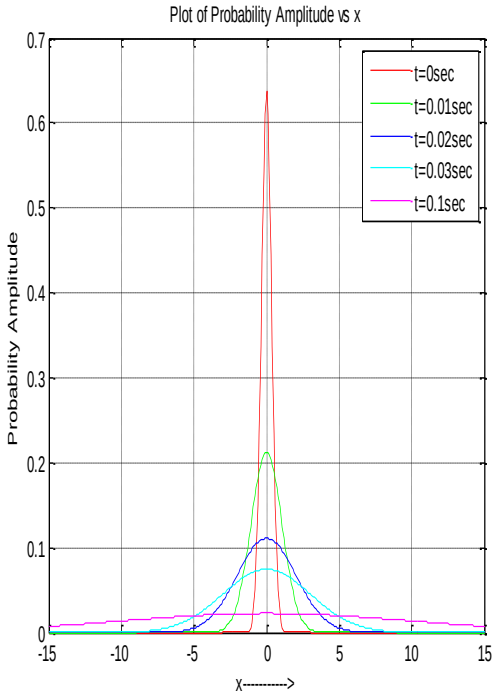

Figure 1: Time Evolution of a Gaussian wave function

2. Square Wave Packet

Consider a square wave packet with amplitude A and width Δx as shown below.

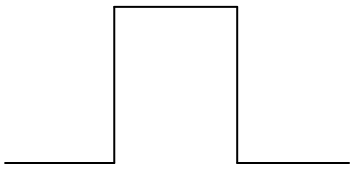

Figure 2: A square wave packet with amplitude A and width Δx

If we try to find its time evolution by the standard method, the integrations encountered are hard to solve. Hence, in order to simplify the calculations, the square wave packet is expressed as a sum of Gaussian wave packets of same width and amplitude as shown below. In order to find the time evolution of the approximated square wave packet, we find the time evolution of the system of Gaussian wave packets. Let us assume that the square wave packet is comprised of three Gaussian wave packets. Let their wave functions be ψ_1, ψ_2 andψ_3; let their mean values be μ_1, μ_2 and $\mu_3$and let

σ be the standard deviation. The forms of ψ at time t=0 and at a later time t for a single Gaussian wave packet is given by equations (1) and (2) respectively. Thus, the square wave packet is expressed as

$$\Psi(x,0) = \psi_1(x,0) + \psi_2(x,0) + \psi_3(x,0)$$

$$\Psi(x,0) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu_1)^2}{2\sigma^2} + ik_0x\right] + \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu_2)^2}{2\sigma^2} + ik_0x\right] + \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu_3)^2}{2\sigma^2} + ik_0x\right]$$

(4)

and at a later time t>0, the square wave packet is expressed as

$$\Psi(x,t) = \psi_1(x,t) + \psi_2(x,t) + \psi_3(x,t)$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}\sigma\left(1+\frac{i\hbar t}{m\sigma^2}\right)^{\frac{1}{2}}} \left\{ exp\left(\frac{2\sigma^2 ik_0x - \frac{i\hbar t{k_0}^2\sigma^2}{m} - \frac{2\hbar tk_0\mu_1}{m}}{2\sigma^2\left(1+\frac{i\hbar t}{m\sigma^2}\right)} \right) + exp\left(\frac{2\sigma^2 ik_0x - \frac{i\hbar t{k_0}^2\sigma^2}{m} - \frac{2\hbar tk_0\mu_2}{m}}{2\sigma^2\left(1+\frac{i\hbar t}{m\sigma^2}\right)} \right) + \right.$$

$$\left. exp\frac{2\sigma 2ik0x - i\hbar tk02\sigma 2m - 2\hbar tk0\mu 3m - x - \mu 322\sigma 21 + i\hbar tm\sigma 2}{} \right\} \qquad (5)$$

The probability distribution P(x,t) of the system of three Gaussian wave packets is given by

$$P(x,t) = |\Psi|^2 = |\psi_1 + \psi_2 + \psi_3|^2$$

$$|\Psi|^2 = {\psi_1}^2 + {\psi_2}^2 + {\psi_3}^2 + 2Re(\psi_1{\psi_2}^*) + 2Re(\psi_1{\psi_3}^*) + 2Re(\psi_2{\psi_3}^*) \qquad (6)$$

where the asterisk indicates complex conjugate. The probability distribution for a single Gaussian wave packet is given by equation (3). Therefore, we can write for ${\psi_1}^2$, ${\psi_2}^2$ and ${\psi_3}^2$ in equation (6) as

$${\psi_1}^2 = \frac{\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)^{-\frac{1}{2}}}{2\pi\sigma^2} exp\left(-\frac{\left\{(x-\mu_1)-\frac{\hbar tk_0}{m}\right\}^2}{\sigma^2\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)} \right)$$

$${\psi_2}^2 = \frac{\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)^{-\frac{1}{2}}}{2\pi\sigma^2} exp\left(-\frac{\left\{(x-\mu_2)-\frac{\hbar tk_0}{m}\right\}^2}{\sigma^2\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)} \right)$$

$$\psi_3{}^2 = \frac{\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)^{-\frac{1}{2}}}{2\pi\sigma^2} exp\left(-\frac{\left\{(x-\mu_3)-\frac{\hbar t k_0}{m}\right\}^2}{\sigma^2\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)}\right)$$

For the term $2Re(\psi_1\psi_2{}^*)$ in equation (6), we write

$$2Re(\psi_1\psi_2{}^*) = 2\frac{\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)^{-\frac{1}{2}}}{2\pi\sigma^2} Re\ exp\left\{\left(\frac{2\sigma^2 ik_0x - \frac{i\hbar t k_0{}^2\sigma^2}{m} - \frac{2\hbar t k_0\mu_1}{m} - (x-\mu_1)^2}{2\sigma^2\left(1+\frac{i\hbar t}{m\sigma^2}\right)}\right) + \left(\frac{-2\sigma^2 ik_0x + \frac{i\hbar t k_0{}^2\sigma^2}{m} - \frac{2\hbar t k_0\mu_2}{m} - (x-\mu_2)^2}{2\sigma^2\left(1-\frac{i\hbar t}{m\sigma^2}\right)}\right)\right\}$$

Simplifying

$$2Re(\psi_1\psi_2{}^*) = \frac{\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)^{-\frac{1}{2}}}{\pi\sigma^2} exp\left\{\left(-\frac{\left((x-\mu_1)-\frac{\hbar k_0 t}{m}\right)^2+\left((x-\mu_2)-\frac{\hbar k_0 t}{m}\right)^2}{2\sigma^2\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)}\right)\right\}$$

Similarly, we can get $2Re(\psi_1\psi_3{}^*)$ and $2Re(\psi_2\psi_3{}^*)$ as

$$2Re(\psi_1\psi_3{}^*) = \frac{\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)^{-\frac{1}{2}}}{\pi\sigma^2} exp\left\{\left(-\frac{\left((x-\mu_1)-\frac{\hbar k_0 t}{m}\right)^2+\left((x-\mu_3)-\frac{\hbar k_0 t}{m}\right)^2}{2\sigma^2\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)}\right)\right\}$$

and,

$$2Re(\psi_2\psi_3{}^*) = \frac{\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)^{-\frac{1}{2}}}{\pi\sigma^2} exp\left\{\left(-\frac{\left((x-\mu_2)-\frac{\hbar k_0 t}{m}\right)^2+\left((x-\mu_3)-\frac{\hbar k_0 t}{m}\right)^2}{2\sigma^2\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)}\right)\right\}$$

We assume k_0 to be zero and use the determined forms of the terms in the LHS of equation (6) and rewrite it in the final form as

$$|\Psi|^2 = \frac{\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)^{-\frac{1}{2}}}{\pi\sigma^2}\left[\frac{1}{2}\left\{exp\left(-\frac{\{(x-\mu_1)\}^2}{\sigma^2\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)}\right)+exp\left(-\frac{\{(x-\mu_2)\}^2}{\sigma^2\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)}\right)+exp\left(-\frac{\{(x-\mu_3)\}^2}{\sigma^2\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)}\right)\right\}+ exp\left(-\frac{(x-\mu_1)^2+(x-\mu_2)^2}{2\sigma^2\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)}\right)+exp\left(-\frac{(x-\mu_1)^2+(x-\mu_3)^2}{2\sigma^2\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)}\right)+exp\left(-\frac{(x-\mu_2)^2+(x-\mu_3)^2}{2\sigma^2\left(1+\left(\frac{\hbar t}{m\sigma^2}\right)^2\right)}\right)\right] \quad (7)$$

For the estimating with three Gaussian wave packets, a square wave function with some arbitrary amplitude within x=0 to x=2 and zero elsewhere was estimated. The Gaussian wave packets had a full width at half maxima equal to $\sigma = 2/10$. The three Gaussian wave packets had mean values at $\mu_1 = 1/3$, $\mu_2 = 1$ and $\mu_3 = 10/6$. For the purpose of approximation, a Gaussian wave packet of the form given in equation (1) was used with $k_0 = 0$. The plot thus obtained is as below

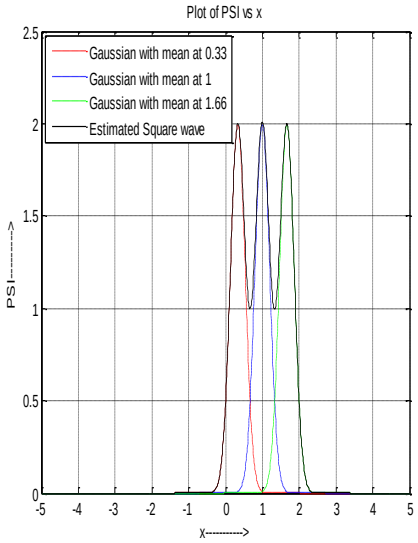

Figure 3: Approximation of a Square wave packet by three Gaussian wave packets

Now, with the same values of mean and standard deviation, the probability distribution of the approximated square wave packet given by equation (7) is plotted against x for different values of time't'. Here, again we use the same values of m and $\hbar$ as in section **1.** The plot obtained is as below.

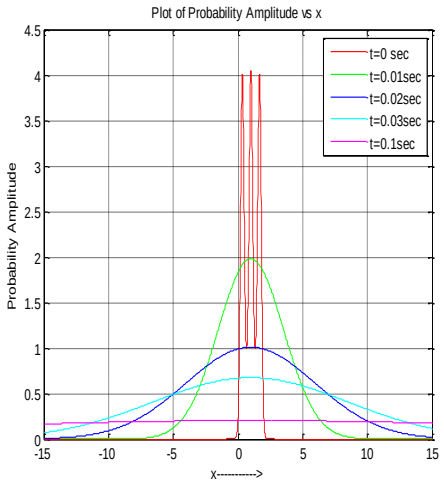

Figure 4: Time Evolution of a Square Wave packet approximated by three Gaussian wave packets

Thus, we see that as the square packet evolves with time, it spreads and approximately becomes a Gaussian.

The square wave packet was also approximated by five Gaussian wave packets with standard deviation $\sigma = 1/5$. The mean values for the Gaussian wave packets were taken to be $\mu_1 = 2/15$, $\mu_2 = 17/30$, $\mu_3 = 1$, $\mu_4 = 43/30$ and $\mu_5 = 28/15$. The width of the square wave packet was fixed to be a=2 and then the interval was divided into 5 parts a/15, 17a/60, a/2, 43a/60 and 14a/15. The resulting figure is shown below.

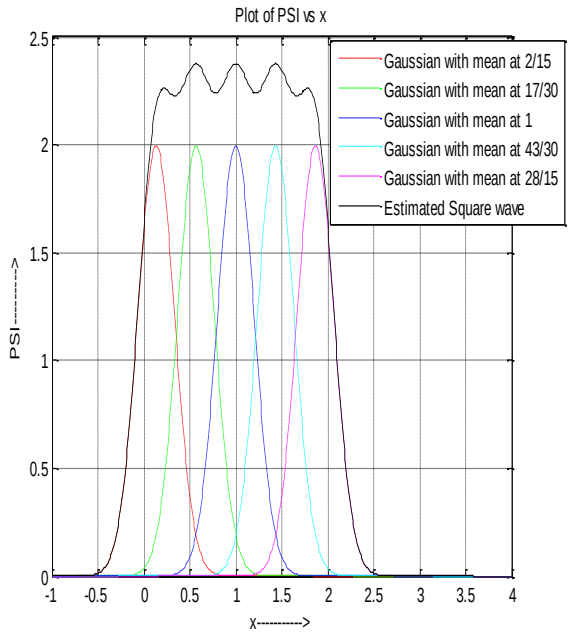

Figure 5: Estimation of Square wave packet by five Gaussian wave packets

The probability distribution of the Gaussian approximation of the square wave packet above is then plotted and is shown below.

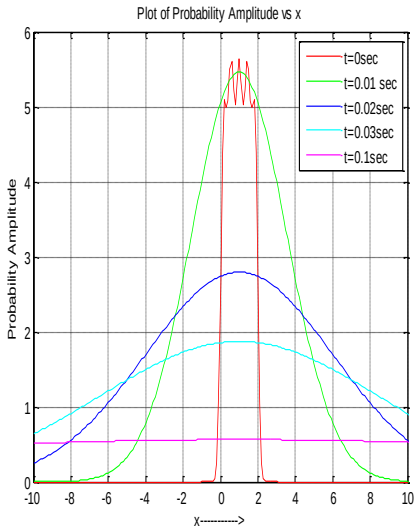

Figure 6: Time evolution of Square wave packet with five Gaussian wave packets

We note that the square wave packet estimated by five Gaussian wave packets gives a better approximation. See section **4** for a detailed discussion.

3. Triangular Wave Packet

In this section, we discuss the time evolution of a triangular wave packet expressed as a sum of Gaussian wave packets. Consider a triangular wave packet which is comprised of three Gaussian wave packets as shown below. Let the wave functions of the three Gaussian wave packets be ψ_1, ψ_2 and ψ_3 and let their mean values be μ_1, μ_2 and μ_3. Let the standard deviations of the three wave functions be σ_1 and σ_2. Here the two Gaussians on either side of the central Gaussian wave have the same standard deviation. The time evolution of a single Gaussian wave packet is given by equation (2). At time t=0, the wave function of the system of triangular wave packet resembles the form of equation (4). At a later time t>0, the wave function of the system is expressed as

$$\Psi(x,t) = \psi_1(x,t) + \psi_2(x,t) + \psi_3(x,t)$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}\sigma_1\left(1+\frac{i\hbar t}{m{\sigma_1}^2}\right)^{\frac{1}{2}}}\left\{exp\left(\frac{2{\sigma_1}^2 ik_0x - \frac{i\hbar t{k_0}^2{\sigma_1}^2}{m} - \frac{2\hbar tk_0\mu_1}{m} - (x-\mu_1)^2}{2{\sigma_1}^2\left(1+\frac{i\hbar t}{m{\sigma_1}^2}\right)}\right)\right.$$

$$\left. + exp\left(\frac{2{\sigma_1}^2 ik_0x - \frac{i\hbar t{k_0}^2{\sigma_1}^2}{m} - \frac{2\hbar tk_0\mu_1}{m} - (x-\mu_3)^2}{2{\sigma_1}^2\left(1+\frac{i\hbar t}{m{\sigma_1}^2}\right)}\right)\right\}$$

$$+ \frac{1}{\sqrt{2\pi}\sigma_2\left(1+\frac{i\hbar t}{m{\sigma_2}^2}\right)^{\frac{1}{2}}} exp\left(\frac{2{\sigma_2}^2 ik_0x - \frac{i\hbar t{k_0}^2{\sigma_2}^2}{m} - \frac{2\hbar tk_0\mu_2}{m} - (x-\mu_2)^2}{2{\sigma_2}^2\left(1+\frac{i\hbar t}{m{\sigma_2}^2}\right)}\right)$$

The probability distribution of the system is given by

$$P(x,t) = |\Psi|^2 = |\psi_1 + \psi_2 + \psi_3|^2$$

$$|\Psi|^2 = {\psi_1}^2 + {\psi_2}^2 + {\psi_3}^2 + 2Re(\psi_1{\psi_2}^*) + 2Re(\psi_1{\psi_3}^*) + 2Re(\psi_2{\psi_3}^*) \qquad (8)$$

The first three terms on the LHS of the above equation are

$${\psi_1}^2 = \frac{\left(1+\left(\frac{\hbar t}{m{\sigma_1}^2}\right)^2\right)^{-\frac{1}{2}}}{2\pi{\sigma_1}^2} exp\left(-\frac{\left\{(x-\mu_1)-\frac{\hbar tk_0}{m}\right\}^2}{{\sigma_1}^2\left(1+\left(\frac{\hbar t}{m{\sigma_1}^2}\right)^2\right)}\right)$$

$${\psi_2}^2 = \frac{\left(1+\left(\frac{\hbar t}{m{\sigma_2}^2}\right)^2\right)^{-\frac{1}{2}}}{2\pi{\sigma_2}^2} exp\left(-\frac{\left\{(x-\mu_2)-\frac{\hbar tk_0}{m}\right\}^2}{{\sigma_2}^2\left(1+\left(\frac{\hbar t}{m{\sigma_2}^2}\right)^2\right)}\right)$$

$${\psi_3}^2 = \frac{\left(1+\left(\frac{\hbar t}{m{\sigma_1}^2}\right)^2\right)^{-\frac{1}{2}}}{2\pi{\sigma_1}^2} exp\left(-\frac{\left\{(x-\mu_3)-\frac{\hbar tk_0}{m}\right\}^2}{{\sigma_1}^2\left(1+\left(\frac{\hbar t}{m{\sigma_1}^2}\right)^2\right)}\right)$$

In the above equation for $\psi_3{}^2$, σ_1 appears as we have assumed that the Gaussians on either side of the central Gaussian gave the same standard deviation i.e. $\sigma_1 = \sigma_3$. Also, the amplitude of ψ_2 depends on the slope of the triangle. The remaining terms in equation (8) are written as

$$2Re(\psi_1\psi_2{}^*) = \frac{\left(1+\frac{\hbar^2t^2}{m^2\sigma_1{}^2\sigma_2{}^2}\right)^{-\frac{1}{2}}}{\pi\sigma_1{}^2\sigma_2{}^2} exp\left[-\frac{(\sigma_1{}^2+\sigma_2{}^2)\left(x-\frac{\hbar tk_0}{m}\right)^2 - 2(\mu_2\sigma_1{}^2+\mu_1\sigma_2{}^2)\left(\frac{\hbar tk_0}{m}-x\right)+\mu_2\sigma_1{}^2+\mu_1\sigma_2{}^2}{2\sigma_1{}^2\sigma_2{}^2\left(1+\frac{\hbar^2t^2}{m^2\sigma_1{}^2\sigma_2{}^2}\right)}\right]$$

And similarly

$$2Re(\psi_1\psi_3{}^*) = \frac{\left(1+\frac{\hbar^2t^2}{m^2\sigma_1{}^2\sigma_3{}^2}\right)^{-\frac{1}{2}}}{\pi\sigma_1{}^2\sigma_3{}^2} exp\left[-\frac{(\sigma_1{}^2+\sigma_3{}^2)\left(x-\frac{\hbar tk_0}{m}\right)^2 - 2(\mu_3\sigma_1{}^2+\mu_1\sigma_3{}^2)\left(\frac{\hbar tk_0}{m}-x\right)+\mu_3\sigma_1{}^2+\mu_1\sigma_3{}^2}{2\sigma_1{}^2\sigma_3{}^2\left(1+\frac{\hbar^2t^2}{m^2\sigma_1{}^2\sigma_3{}^2}\right)}\right]$$

$$2Re(\psi_2\psi_3{}^*) = \frac{\left(1+\frac{\hbar^2t^2}{m^2\sigma_3{}^2\sigma_2{}^2}\right)^{-\frac{1}{2}}}{\pi\sigma_3{}^2\sigma_2{}^2} exp\left[-\frac{(\sigma_3{}^2+\sigma_2{}^2)\left(x-\frac{\hbar tk_0}{m}\right)^2 - 2(\mu_2\sigma_3{}^2+\mu_3\sigma_2{}^2)\left(\frac{\hbar tk_0}{m}-x\right)+\mu_2\sigma_3{}^2+\mu_3\sigma_2{}^2}{2\sigma_3{}^2\sigma_2{}^2\left(1+\frac{\hbar^2t^2}{m^2\sigma_3{}^2\sigma_2{}^2}\right)}\right]$$

Once again, we assume the value of k_0 to be zero and since we have also assumed $\sigma_1 = \sigma_3$, we rewrite equation (8) in the final form as

$$\Psi^2 = \frac{\left(1+\left(\frac{\hbar t}{m{\sigma_1}^2}\right)^2\right)^{-\frac{1}{2}}}{2\pi{\sigma_1}^2} exp\left(-\frac{\{(x-\mu_1)\}^2}{{\sigma_1}^2\left(1+\left(\frac{\hbar t}{m{\sigma_1}^2}\right)^2\right)}\right) + \frac{\left(1+\left(\frac{\hbar t}{m{\sigma_2}^2}\right)^2\right)^{-\frac{1}{2}}}{2\pi{\sigma_2}^2} exp\left(-\frac{\{(x-\mu_2)\}^2}{{\sigma_2}^2\left(1+\left(\frac{\hbar t}{m{\sigma_2}^2}\right)^2\right)}\right)$$

$$+ \frac{\left(1+\left(\frac{\hbar t}{m{\sigma_1}^2}\right)^2\right)^{-\frac{1}{2}}}{2\pi{\sigma_1}^2} exp\left(-\frac{\{(x-\mu_3)\}^2}{{\sigma_1}^2\left(1+\left(\frac{\hbar t}{m{\sigma_1}^2}\right)^2\right)}\right)$$

$$+ \frac{\left(1+\frac{\hbar^2 t^2}{m^2{\sigma_1}^2{\sigma_2}^2}\right)^{-\frac{1}{2}}}{\pi{\sigma_1}^2{\sigma_2}^2} exp\left[-\frac{({\sigma_1}^2+{\sigma_2}^2)x^2 + 2(\mu_2{\sigma_1}^2+\mu_1{\sigma_2}^2)x + \mu_2{\sigma_1}^2+\mu_1{\sigma_2}^2}{2{\sigma_1}^2{\sigma_2}^2\left(1+\frac{\hbar^2 t^2}{m^2{\sigma_1}^2{\sigma_2}^2}\right)}\right]$$

$$+ \frac{\left(1+\frac{\hbar^2 t^2}{m^2{\sigma_1}^2{\sigma_1}^2}\right)^{-\frac{1}{2}}}{\pi{\sigma_1}^2{\sigma_1}^2} exp\left[-\frac{({\sigma_1}^2+{\sigma_1}^2)x^2 + 2(\mu_3{\sigma_1}^2+\mu_1{\sigma_1}^2)x + \mu_3{\sigma_1}^2+\mu_1{\sigma_1}^2}{2{\sigma_1}^2{\sigma_1}^2\left(1+\frac{\hbar^2 t^2}{m^2{\sigma_1}^2{\sigma_1}^2}\right)}\right]$$

$$+ \frac{\left(1+\frac{\hbar^2 t^2}{m^2{\sigma_1}^2{\sigma_2}^2}\right)^{-\frac{1}{2}}}{\pi{\sigma_1}^2{\sigma_2}^2} exp\left[-\frac{({\sigma_1}^2+{\sigma_2}^2)x^2 + 2(\mu_2{\sigma_1}^2+\mu_3{\sigma_2}^2)x + \mu_2{\sigma_1}^2+\mu_3{\sigma_2}^2}{2{\sigma_1}^2{\sigma_2}^2\left(1+\frac{\hbar^2 t^2}{m^2{\sigma_1}^2{\sigma_2}^2}\right)}\right]$$

In our numerical analysis, the standard deviation of the central Gaussian wave packet i.e. σ_2 was chosen to be 0.5 and σ_1 and σ_3 were chosen to be 1. The mean values were chosen to be -5/2.5, 0 and 5/2.5. The probability distribution of the approximated triangular wave packet is plotted against x for different values of time t. The values of m and $\hbar$ were the same as the ones used in sections **1** and **3.**

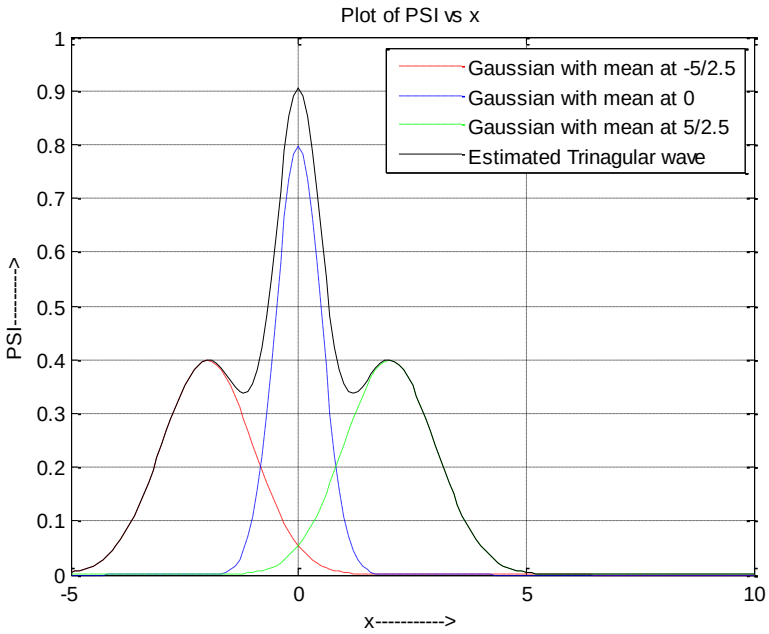

Figure 7: Estimation of Triangular wave packet by three Gaussian wave packets

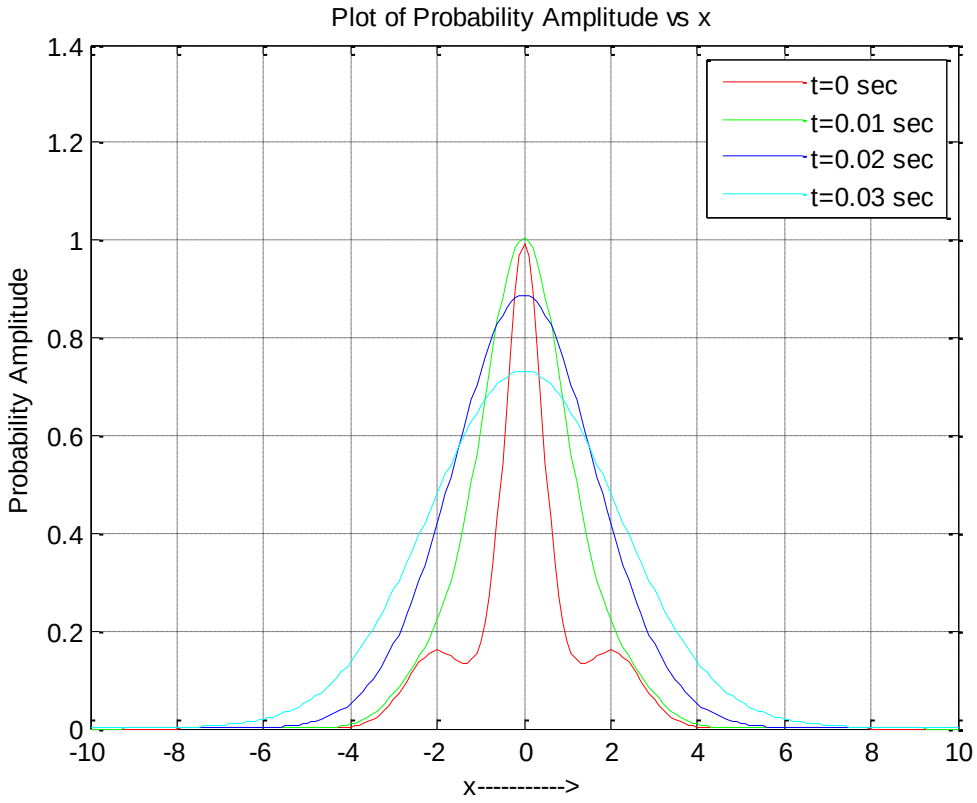

Figure 8: Time Evolution of Triangular wave approximated by 3 Gaussians

4. Results and Discussion

When the Square wave packet was approximated with three Gaussians, its amplitude was found to be 2 (fig. 4). Here the estimated square wave consisted of dips where the overlapping of Gaussian waves was not as significant. This is due to the intermediate terms in equation (7). Its probability amplitude was doubled, i.e. approx. 4 at time t=0, which is as expected. The probability amplitude at time t=0 consisted of some irregularities in the peak. With time, as the Gaussian waves evolved, so did the estimated square wave and thus the probability distribution of the square wave became smoother, broader and assumed an almost Gaussian shape which is clearly visible in fig. 5. In the case of Square wave approximated with five Gaussians, the dip in the final wave form reduced considerably due to significant overlapping thus estimating the square wave packet better than the one with the three Gaussian wave packets. Thus we see that the accuracy increases when the number of wave packets is increased. The probability distributions at t=0 and at later times are assumed to have the same behavior as mentioned earlier.

Similar results were obtained in the case of a Triangular wave packet being approximated by three Gaussian wave packets.

In the method adopted, care must be taken while choosing the standard deviation and mean values for the approximating Gaussian waves. The advantage of this method is that it is applicable for any wave packet in principle. The number of iterations can be improved by computer programming since there is an increase in the number of wave functions and also the number of times they are added

5. Acknowledgements

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Revisiting Fizeau's Observations: Spectral study of Na source using Newton's rings

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Abstract

Conventional Newton's rings experiment is generally used in undergraduate laboratory to determine the average wavelength of Na doublet. A minor modification of this setup enables us to view simultaneously Newton's rings both in reflected as well as transmitted light. A movable glass plate with respect to the plano-convex lens allows us to observe the variation of contrast/visibility of these fringes and thus allows us to determine the separation of the Na yellow doublet.

1. Introduction

Newton's rings experiment is a classic example of fringes of equal thickness or Fizeau fringes[1,2]. This is generally used in undergraduate optics laboratory, as it enhances conceptual understanding of internal and external reflections and associated phase shifts. It also requires fair degree of experimental skill to perform this. Even in the literature [3,4,5,6] we find it being used for classroom demonstrations, for determination of wavelength of He-Ne laser etc. The apparatus can be rearranged by mounting the plano-convex lens and the glass plate vertically. This modification lends itself to observe the Newton's rings in the transmitted light. Moreover, if the glass plate is made movable, as was done in this work, this could also be used to estimate the Na doublet separation apart from being used to determine the average wavelength.

2. Theory of formation of Newton's rings

For a plane parallel glass plate of thickness, t, the path difference between two partially reflected light from top and the bottom surface of plate is given by [7]:

$$\delta = 2\, n_f t \cos(\theta_t) \qquad (1)$$

where n_f :Refractive index of glass plate, t:Thickness of glass plate, θ_t:Angle of refraction

If the glass plate is of varying thickness t, the optical path difference varies even without variation in the angle of incidence. Thus if the direction of the incident beam is fixed, say at normal incidence, a dark or bright fringe will be associated with a particular thickness for which λ satisfies the condition for destructive or constructive

interference, respectively. For this reason, fringes produced by a plate of variable thickness are called Fringes of equal thickness or Fizeau fringes. For a monochromatic and collimated light, it is only the thickness variation which can result in path difference variation. Therefore, Fizeau fringes are contours of constant thickness.

When an air film, formed between the spherical surface of plano-convex lens and an optical flat is illuminated from a laser or sodium vapour lamp, equal thickness contours for a perfectly spherical surface are formed. These circular fringes are called Newton's rings. These fringes are formed around the point of contact. At the centre, thickness of the air film is zero and hence the path difference between the two reflected rays is zero due to propagation. But the central fringe is dark as a consequence of phase shift of π due to one external reflection.

3. Finding the Fine structure using Newton's rings

We know that the sodium lamp emits two wavelengths: 5890A and 5896A. These two wavelengths form two sets of Newton's rings with coincident centres. If we examine a few rings near the point of contact of lens and glass plate, the two sets of rings appear to coincide; but if they are traced to a sufficient distance from the centre, the misalignment becomes more and more apparent. Consequently, after some distance, the bright fringe of one set of rings will occupy the same position as the dark fringe of the other set, and they will mutually annihilate to a uniform intensity. If on the other hand, the glass plate is moved away from the lens, mutual annihilation would take place at the centre itself satisfying equation of the type (2) where d is the distance between the glass plate and plano-convex lens. Continuing the same line of reasoning, it is evident that perfect coincidence and perfect misalignment of the two systems of rings would recur alternately at regular intervals[8].

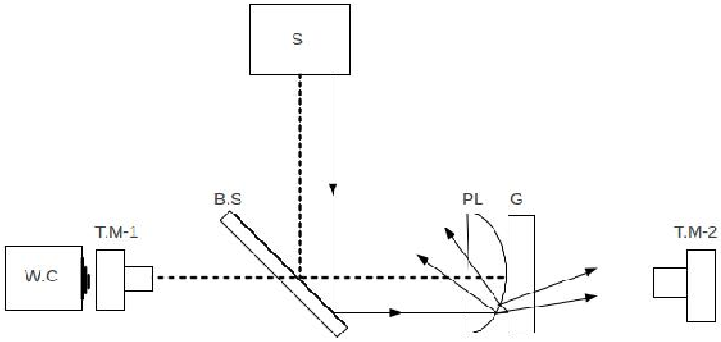
$$2d = m_1\lambda_1 = (m_2 + ½)\,\lambda_2 \qquad (2)$$

To observe this variation of contrast, the glass plate is made movable in a perpendicular direction with respect to plano-convex lens. When the glass plate is moved away, the higher order fringes appear at the centre. Due to finite coherence length of source, the contrast variation can be clearly seen as the glass plate is moved.

The separation of two wavelengths in the Na doubletthat is used to observe interference is related to the path difference, as in the case of Michelson interferometer[7]. If $\Delta\lambda$ is the separation between two spectral lines and λ is the average wavelength, then

$$\Delta\lambda = \frac{\lambda^2}{2\,\Delta d} \qquad (3)$$

whereΔd is the mirror movement required between two consecutive coincidences(bright fringe of λ_1 overlapping with bright fringe of λ_2) in a Michelson interferometer. This formula can be used in Newton's rings in the case where glassplate is moved with respect to the plano-convex lens. In this case, Δd represents distance through which the glass plate is moved from zero pathdifference(glass plate touching the plano-convex lens) position to the subsequent region of maximum contrast.



Fig(1) : Schematic diagram of Newton's rings with glass plate, G and plano-convex lens, PL mounted vertically and the Glass plate made movable. T.M-1 and T.M-2: Travelling microscopes, W.C: Webcam connected to travelling microscope, BS: Beam splitter, S: Light source

4. Experimental details and Results

4.1 Determination of average wavelength of Na

The average wavelength of Na light is easily determined by measuring the diameters of a few rings. A plot of radius-square and the ring number gives a straight line. By finding the slopeand knowing the radius of curvature,R, we can get the wavelength of Na light using the formula,

$$\lambda = \frac{r_{m+p}^2 - r_m^2}{pR} \qquad (4)$$

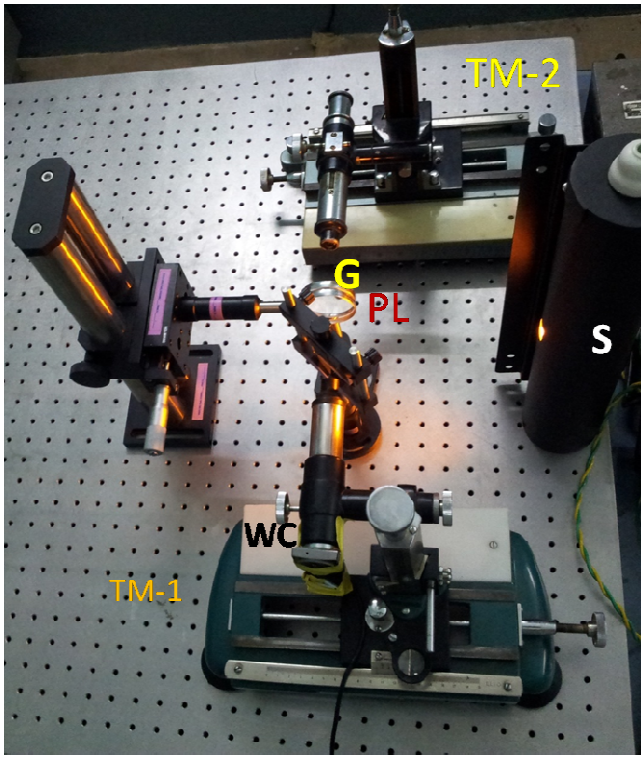
where, r_m and r_{m+p} are radii of the m^{th} ring and $(m+p)^{th}$ ring, R is Radius of curvature of the lens.

The conventional Newton's rings experiment was performed and the graph of ring no. m versus $r_m^2$was plotted as shown in Fig(3). The slope of this graph is $(r_{m+p}^2 - r_m^2)/p$. Therefore the average wavelength of sodium source

= slope /radius of curvature (R)

= 0.297 x10^{-6} m^2 / 0.5 m

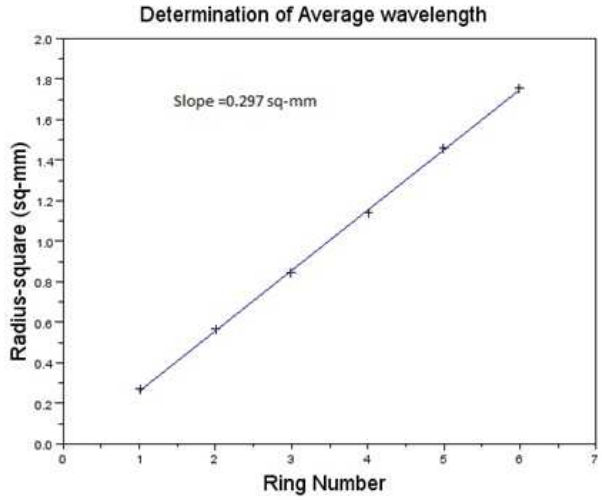
= 595nm



Fig(2): Photograph of experimental setup showing the vertical mounting of plano-convex lens, PL and glass plate, G

The expected value of average wavelength of Na is 589.3nm and the experimental value is accurate to about one percent. The average value is

required to determine the separation of the Na doublet.



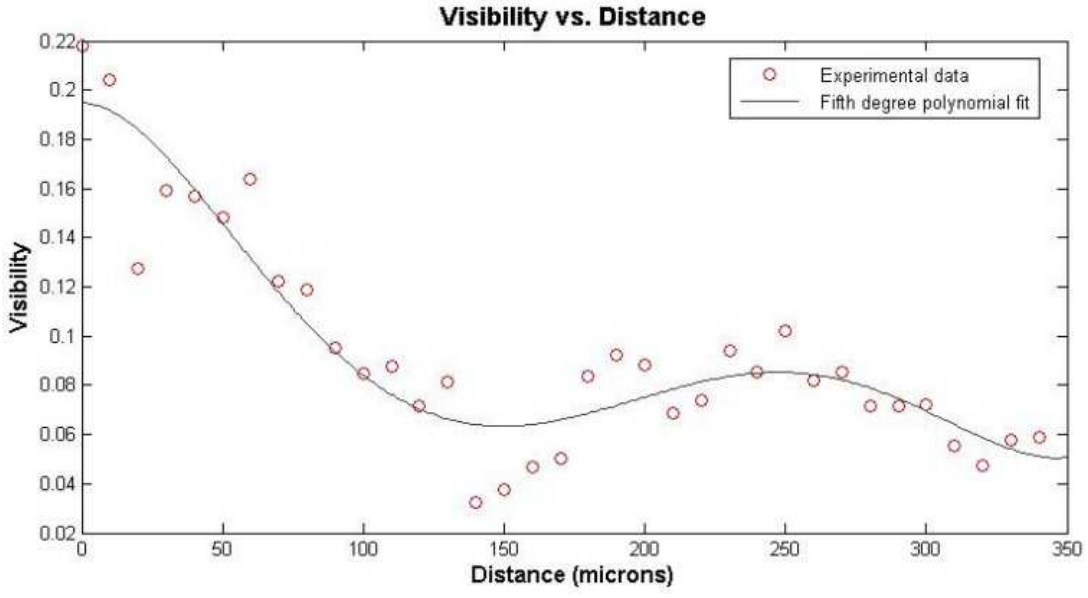
Fig(3) : Graph of ring no versus diameter-square to determine the average wavelength. The linear regression is used to get the best-fit straight line, with a correlation coefficient of 0.99

4.2 Estimation of Na doublet separation

The schematic diagram and the photograph of the experimental setup are shown in fig(1) and fig(2) respectively. The glass plate was mounted on a micrometer translation stage so that its distance can be varied in a controlled manner. Newton's rings in the reflected light were captured using a webcam (without the lens) attached to travelling microscope without the eyepiece. Newton's rings were captured for every 10µm distance. From the snapshot, the visibility of the fringes was calculated using ImageJsoftware (freely downloadable, image processing software developed by National Institutes of Health).Sample snapshots at different distances are provided in fig(5) to fig(9) which clearly show the contrast variation. Fig(10) & fig(11) show the Newton's rings in the reflected and transmitted light.Plot of distance versus visibility is shown in the fig(4). As expected, there is a periodic variation of contrast starting with its maximum value at zero path difference. From the graph, we get $\Delta d = 250\mu m$, which is the separation between two successive maxima in the graph.

$$\Delta\lambda = \frac{\lambda^2}{2\,\Delta d} = (595nm)^2/2(250\mu m) = 7\ A$$

This shows that the separation of the Na doublet whose expected value is 6A, could be estimated to an accuracy of about 17 percent using this analysis.



Fig(4) : Plot of visibility as a function of distance of glass plate from the plano-convex lens.

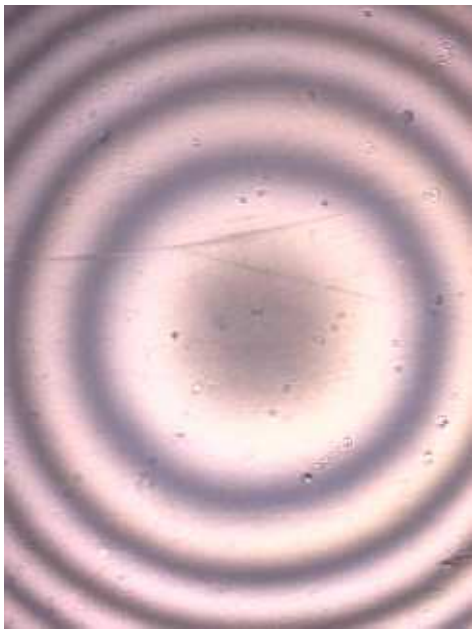


Fig. (5): Zero path difference

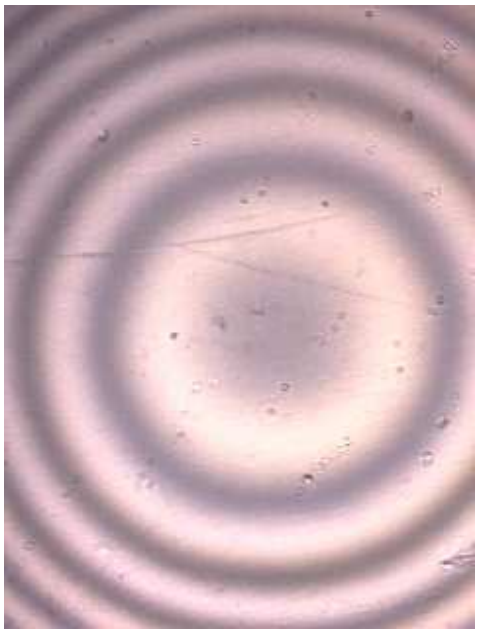


Fig. (6): Path difference = 50micron

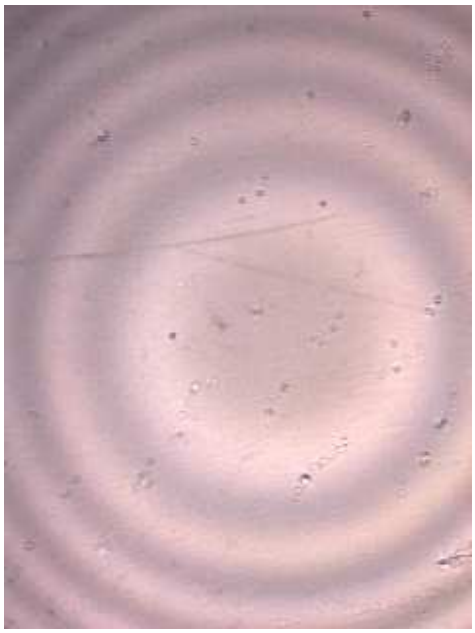


Fig. (7): Path difference=100 micron

Fig. (8): Path difference=150 micron

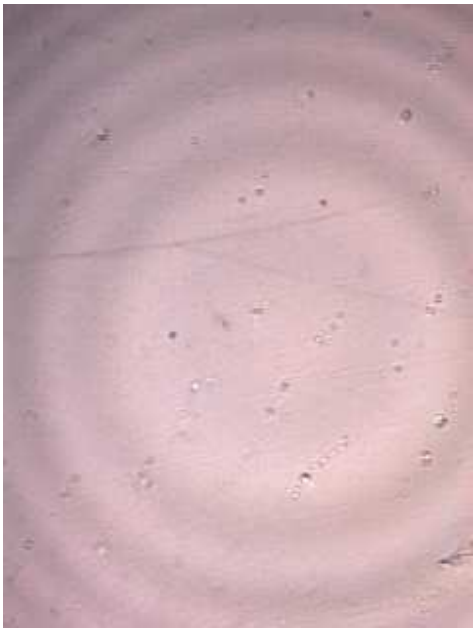


Fig.(9): Path difference=250 micron

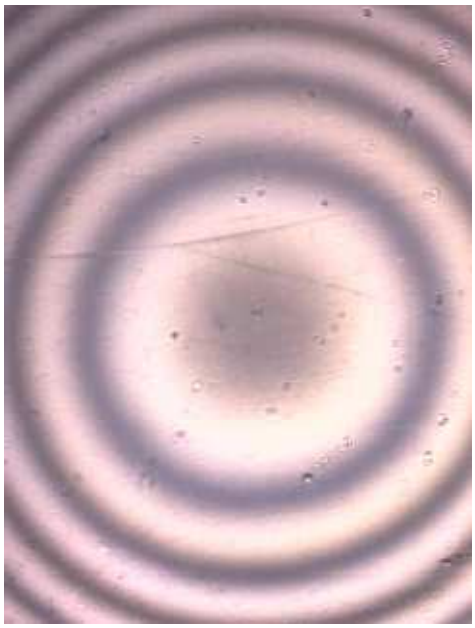


Fig. (10): Newton's rings: Reflected light

- **Central Dark Fringe**
- **High Contrast**

Fig. (11): Newton's rings: Transmitted light

- **Central Bright Fringe**
- **Low Contrast**

5. Conclusion

In this work, we have revisited Fizeau's observations by modifying the conventional Newton's rings setup. Both the glass plate and plano-convex lens are mounted vertically which enables us to view both the reflected and transmitted rings simultaneously, compare and contrast them with considerable ease. Moreover, the glass plate mounted on a micrometer translation stage, allows us to observe clearly the variation fringe contrast/visibility. From the graph of distance versus visibility, we were able to calculate the separation of Na doublet. This also enables us to make an estimate of coherence length of Na source by noting the distance to the first minimum in the graph.

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Functional differential equations. 3: Radiative damping

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Abstract

What is the solution of the equation of motion of a single classical charged particle with radiative damping? Contrary to the physical expectation, the mathematical solution is anti-damped! Attempts to curb these runaway solutions lead to pre-acceleration. Worse, despite a century of effort, there is still no way to obtain a proper solution in a general context. This failure of classical electrodynamics is intrinsic, irrespective of the hydrogen atom, and hence needs to be remedied. We outline a general method to resolve the infinities of quantum electrodynamics (renormalization problem). The same method was recently applied to resolve the infinities of classical electrodynamics. This involves a modification of Maxwell's equations at the microphysical level. The resulting equations of motion of even a single charged particle with radiative damping are functional differential equations (FDEs). These FDEs can and have been solved. The implications for quantum mechanics are postponed to the next article.

1 Recap

In two earlier articles[1, 2] in this series, we saw that functional differential equations (FDEs) are fundamentally different from ordinary differential equations (ODEs). Hence, doing physics with FDEs leads to a paradigm shift in physics. To solve retarded FDEs, for example, we need to specify past history, not initial data alone, as is the case with the ODEs of Newtonian mechanics. Again, with FDEs, volume in phase space is *not*, in general, preserved (so fine-grained entropy does *not* stay constant), so we must reconsider sta-

tistical mechanics. These significant qualitative differences between FDEs and ODEs also mean that FDEs cannot, in general, be validly approximated by ODEs.

We also saw that FDEs arise naturally in physics, so that this new physics does *not* involve any new physical hypothesis, but is a matter of doing the math right. That is because FDEs are equivalent to a *coupled* system of ODEs and PDEs. Hence, FDEs arise naturally in the context of the electrodynamic 2-body problem, which involves a coupling between the Heaviside-Lorenz force law (ODEs) (according to which each particle moves) and Maxwell's equations which are partial differential equations (PDEs) (according to which each particle acts on the other).

This understanding of FDEs also explains the need for past data. To solve Maxwell's equations we need to specify the appropriate Cauchy data, which is the counterpart of initial data for PDE. That is, we need to specify the electric and magnetic fields on a hypersurface (i.e., at an "instant" of time). If we use retarded Green functions, actually specifying these fields on an entire hypersurface requires data for the entire past world lines of the particles which produce those fields. This requirement of past data, subtly hidden by the field picture, is only made manifest by using the particle pictures and FDEs.

How exactly does this affect electrodynamics? To this end we re-examined the question of the classical hydrogen atom. Physicists are taught in high-school that classical electrodynamics cannot describe the hydrogen atom. The argument for this proceeds as follows. It first supposes that in the *absence* of radiation damping, central orbits are stable for the electrodynamic two body problem. It then concludes, heuristically, that due to radiation damping those orbits are actually unstable.

With our new understanding of FDEs it is clear that this conclusion is based on faulty reasoning. The Coulomb force does not equal the full electrodynamic force. The full electrodynamic force leads to FDEs, so approximating it by the Coulomb force involves approximating FDEs by ODEs, a process known to be incorrect in general. Therefore, the claim that central orbits are stable in the absence of radiation damping was never properly established.

The first actual solution of the FDEs of the electrodynamic 2-body problem, with the full electrodynamic force, was carried out by this author only in 2004.[3] It showed that the solution with the Coulomb force is, in fact, incorrect. Heuristically, we observed that retardation leads to a delay torque, so that an electron tends to fall out of the atom, in the absence of radiation damping.

We concluded with the natural question: what happens in the presence of radiation damping? Are there motions (not necessarily circular orbits) for which the delay torque and the radiation damping cancel (either exactly or on an average)?

2 The problem of radiative damping

2.1 The formula for radiative damping

A peculiar difficulty arises in trying to give a rigorous answer to this question. First, we obviously need a quantitative account of radiation damping. Now, standard physics texts (e.g., [4], equation 11.80, p. 467) give a formula (Abraham-Lorentz formula) for the force due to radiative damping

$$\boldsymbol{F}_{\rm rad} = \frac{\mu_0 q^2}{6\pi c}\dot{\boldsymbol{a}}, \tag{1}$$

where μ_0 is the permeability of free space, q is the charge, c is the speed of light, $\boldsymbol{a}$ is the acceleration of the charge, and, as before, dots denote derivatives with respect to time. The physical understanding of this formula is that an accelerating charge radiates energy, and therefore its motion must be damped. This formula describes the self-force on the electron responsible for the damping.

2.2 The equation of motion of a charge

But this force has a peculiarity: it depends upon the derivative or rate of change of *acceleration*. Thus, the equation of motion of an accelerated charged particle under the influence of an external force $\boldsymbol{F}_{\rm ext}$ is

$$\begin{aligned} m\boldsymbol{a} &= \boldsymbol{F}\text{ext} + \boldsymbol{F}_{\rm rad} \\ &= \boldsymbol{F}\text{ext} + \frac{\mu_0 q^2}{6\pi c}\dot{\boldsymbol{a}}. \end{aligned} \tag{2}$$

Because of the appearance of $\dot{\boldsymbol{a}} = \dddot{\boldsymbol{x}}$ this is a *third* order ODE, unlike the ODEs of classical mechanics which are all of second order. Hence, to solve for the motion of a single charged particle, one must now prescribe also $\dot{\boldsymbol{a}}(0)$ or the initial acceleration of the charge.

2.3 The runaway solutions

It is not clear on what principles the specification of $\boldsymbol{a}(0)$ would be based, but let us see what happens in the simplest case. That is we consider a particle moving in one dimension, without any external forces, so that $\boldsymbol{F}_{\rm ext} = 0$. In this case the equation (2) can be rewritten as the simple equation

$$\tau\dot{a} = a, \tag{3}$$

where

$$\tau = \frac{\mu_0 q^2}{6\pi mc}. \tag{4}$$

The equation (3) has an equally simple solution

$$a(t) = a_0 e^{\frac{t}{\tau}}, \tag{5}$$

where a_0 is the initial acceleration.

But this is catastrophic! It is evident that so long as $a_0 \neq 0$, no matter what its value, the acceleration of the particle, a, *increases* exponentially. Thus, an arbitrary non-zero initial acceleration blows up, so these are known as runaway solutions. Under its own self-action, due to radiation damping, the particle continuously accelerates! Instead of damping (as expected on physical grounds), the mathematics tells us what we have here is unbounded anti-damping!

This difficulty has been known for so long that physicists have become a bit blasé about it, and easily satisfied with various partial solutions which have been offered. To understand these, let us first find a general solution of (2).

2.4 General solution of the equations of motion

To this end, let us rewrite (2) as

$$m(\boldsymbol{a} - \tau\dot{\boldsymbol{a}}) = \boldsymbol{F}_{\text{ext}}. \tag{6}$$

To help solve the equations, we introduce a new variable $\boldsymbol{a}_1(t)$ by

$$\boldsymbol{a}_1(t) = e^{-\frac{t}{\tau}}\boldsymbol{a}(t) \quad \text{or} \quad \boldsymbol{a} = e^{\frac{t}{\tau}}\boldsymbol{a}_1(t). \tag{7}$$

Then

$$\dot{\boldsymbol{a}} = \frac{1}{\tau}e^{\frac{t}{\tau}}\boldsymbol{a}_1 + e^{\frac{t}{\tau}}\dot{\boldsymbol{a}}_1. \tag{8}$$

Hence,

$$\boldsymbol{a} - \tau\dot{\boldsymbol{a}} = -\tau e^{\frac{t}{\tau}}\dot{\boldsymbol{a}}_1, \tag{9}$$

so that (6) can be rewritten

$$\dot{\boldsymbol{a}}_1 = -\frac{1}{m\tau}e^{-\frac{t}{\tau}}\boldsymbol{F}_{\text{ext}}. \tag{10}$$

Equation (10) can be solved just by integrating it.

$$\boldsymbol{a}_1(t) = -\frac{1}{m\tau}\int_0^t e^{-\frac{t'}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt' + \boldsymbol{a}_1(0), \tag{11}$$

where $\boldsymbol{a}_1(0)$ is a constant of integration or the initial value. Note that $\boldsymbol{a}_1(0) = \boldsymbol{a}(0)$ by the definition (7). Hence, the general solution of (6) can be rewritten

$$\boldsymbol{a}(t) = \boldsymbol{a}(0)e^{\frac{t}{\tau}} - \frac{1}{m\tau}\int_0^t e^{\frac{(t-t')}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt'. \tag{12}$$

2.5 Dirac's proposal

Now, nothing in earlier physics tells us what principles we should use to fix the value of $\boldsymbol{a}(0)$. Therefore, Dirac[5] in 1938 suggested we should fix it by the formula

$$\boldsymbol{a}(0) = \frac{1}{m\tau}\int_0^\infty e^{-\frac{t'}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt'. \tag{13}$$

To see the point of this, let us plug in this value of $\boldsymbol{a}(0)$ into (12). We now obtain

$$\begin{aligned}
\boldsymbol{a}(t) &= e^{\frac{t}{\tau}}\left(\boldsymbol{a}(0) - \frac{1}{m\tau}\int_0^t e^{-\frac{t'}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt'\right) \\
&= \frac{e^{\frac{t}{\tau}}}{m\tau}\left(\int_0^\infty e^{-\frac{t'}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt' - \right. \\
&\qquad \left.\int_0^t e^{-\frac{t'}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt'\right) \\
&= \frac{e^{\frac{t}{\tau}}}{m\tau}\left(\int_t^\infty e^{-\frac{t'}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt'\right) \\
&= \frac{1}{m\tau}\left(\int_t^\infty e^{\frac{(t-t')}{\tau}}\boldsymbol{F}_{\text{ext}}(t')dt'\right).
\end{aligned} \tag{14}$$

Since $(t - t') < 0$ for $t' \in (t, \infty)$, the integral will converge for any reasonable external force described by an integrable (or even slowly increasing) function $\boldsymbol{F}_{\text{ext}}$. We can see this more clearly, by making the change of variables $t' = t + \tau s$, to rewrite (14) as

$$\boldsymbol{a}(t) = \frac{1}{m\tau}\int_0^\infty e^{-s}\boldsymbol{F}_{\text{ext}}(t + \tau s)ds. \tag{15}$$

Hence, the acceleration remains finite for all time.

2.6 Pre-acceleration

However, Dirac's proposal has a peculiar side-effect. It is clear from (15) that the accelera-

tion at time t is decided by a weighted average over all *future* forces. Hence, if, for example, an impulsive force is applied to a charged particle, it would start moving *before* the force is applied.

To see this, consider the case of 1-dimensional motion, and suppose a δ function force is applied at time $t = 0$. That is, $F_{\text{ext}}(t) = \delta(t)$. For the integral in (15) to be non-zero, we must have $t + \tau s = 0$, and this is possible only for $t < 0$. So, the solution is

$$a(t) = \begin{cases} \frac{1}{m\tau} e^{\frac{t}{\tau}} & \text{if } t < 0 \\ 0 & \text{if } t \geq 0. \end{cases} \qquad (16)$$

That is, the particle accelerates *before* the force is applied, and stops accelerating when it is applied. Hence, this is called pre-acceleration. This is considered "unphysical" since non-causal.

The usual defence is that the "violation" of causality takes place over a small time. For the case of an electron, the constant τ in (4) may be rewritten as

$$\tau = \frac{2}{3} \cdot \frac{q^2}{4\pi\epsilon_0 mc^3} = \frac{2}{3}\frac{r_e}{c}, \qquad (17)$$

where r_e is the classical radius of the electron, so that τ is of the order of the time it takes for light to traverse the classical radius of the electron; $\tau \approx 10^{-23}$s is also called the relaxation time of the electron. So, the argument is that the violation of causality takes place over such small times that it is of no consequence.

2.7 Discussion

Now, I have been advocating the rejection of perfect "causality" for over 35 years,[6] and for the last 20 years I have been arguing that rejecting (mechanistic) "causality" in physics is a matter of elementary commonsense,[7] and the only way to explain mundane "causality" or the mundane experience of billions of people repeated thousands of time each day. So I cannot regard the failure of (mechanistic)[1] "causality" as some kind of a "violation" to be concerned about.

However, the real problem with the above solution is this: the formula does not do its basic job. It is small consolation to know that the acceleration is finite, because we cannot calculate its value! This matter has been subject to a long debate, and several variants on Dirac's technique and the Lorentz-Dirac equation have been proposed. But this objection also applies to other proposed "solutions" such as the one suggested by Plass:[9] the initial or boundary values are required to hold *exactly*. The slightest variation from it restores the runaways, so these methods of supposedly taming the infinities of classical electrodynamics are impractical for the problem at hand which is this: to calculate the motion of the electron in a hydrogen atom with radiative damping.

[1]The term "causality" is vague and can have diametrically opposite meanings, which are often confounded in philosophical discussions. Specifically, mechanistic "causality" as used by physicists as physicists means the exact opposite of mundane "causality", as used by physicists as human beings! Conflating these two causes great confusion see, e.g., [8].

A simple-minded way out is to say that Maxwell's equations anyway fail for the hydrogen atom where quantum mechanics applies. However, the failure of Maxwellian electrodynamics, as analysed above, is intrinsic, and makes no reference to the hydrogen atom. Therefore, we need to understand why this celebrated Maxwellian theory, which is otherwise useful, fails so miserably that it cannot describe the motion of even a single electron! More pragmatically, understanding the reasons for the intrinsic failure of Maxwell's equations may suggest an appropriate correction which opens the path to a fresh understanding of quantum mechanics, which is needed to resolve the problems facing quantum computing today.

Therefore, let us press on with our enquiry into the infinities which arise in the motion of a single charged particle in Maxwellian electrodynamics, their origin and their resolution.

2.8 The finite size electron

The most obvious suspicion is that these infinities have something to do with the assumption that charged particles must be like idealised geometric points. Radiation damping is attributed to the self-action of a charge. In the field picture, that self-action is described by a charge interacting with its own field. That field, however, blows up at the position of the particle if that is assumed to be a point.

The next obvious step is to suppose that the point-charge description is a simplification, and what we really have is a finite distribution of charge. That, in fact, was the first proposed solution to this problem, ironically proposed by Lorentz.

However, this notion of a finite-size electron encountered several serious problems. In the first place, suppose we simply smear out the electron charge over a sphere or shell. The Coulomb repulsion of one part of the charge distribution acting on another would blow apart the charge distribution. What holds it together?

One could get around this problem by postulating some new forces which hold the electron together. There is, however, a far more serious problem with this solution: it is not Lorentz invariant. We can hardly abandon Lorentz invariance because the requirement of Lorentz-invariance is tied to the current definition of time measurement, as I have explained in my book[10] and in an earlier article in this journal, and we cannot do any physics without a way to measure time. A finite distribution of charge cannot easily be described in a Lorentz invariant way. A sphere in one frame would not remain a sphere in another, for the Lorentz transform distorts a sphere into an ellipsoid. The problem of a Lorentz invariant or covariant extended electron has resisted attempts at a solution for the past century.

2.9 Is the limiting procedure valid?

We could get around this problem too, but there is another subtle problem which has not been noticed, but is rather serious. The usual

derivation of the third order radiation reaction force (1) does start by assuming a finite distribution of charge (for example, [4] starts with a dumb-bell charge distribution). Since, however, this finite charge distribution cannot be described in a Lorentz invariant way, the usual derivation proceeds to the limit of a point charge. The problem of Lorentz invariance disappears in the limit.

But is this limiting procedure valid? The question was first raised by me long ago, in this very journal.[11] The doubt about the validity of the limiting procedure may be explained in simple terms as follows. In a finite charge distribution, when one part of the charge distribution acts on another, there is a retardation or delay involved. Therefore, the equations involved are FDEs; we have seen that. However, when we proceed to the limit of a point charge, the final equation of motion with radiative damping is just an ODE, as above.

So, mathematically, the limiting procedure amounts to "Taylor" expanding in powers of the delay, and then proceeding to the limit as the delay goes to zero. This limiting process converts an FDE into a higher-order ODE. We have seen[1] that this is an incorrect procedure, therefore the limiting process is not valid, even though it looks plausible, and is followed by all texts in electrodynamics today! Thus, there is a fundamental problem concerning the derivation of very formula for radiation damping (1).

3 Infinities of classical and quantum electrodynamics

3.1 A connection?

How to correct the derivation of radiation reaction? Long ago, Wheeler and Feynman thought that the infinities of quantum electrodynamics might be fixed by correcting the corresponding infinities in classical electrodynamics. Today, physicists believe that the infinities in quantum electrodynamics have been fixed through what is called renormalization. But the infinities of classical electrodynamics still stay unfixed! Nevertheless, the hunch of a connection between the two infinites was right.

Thus, long ago, I suggested a novel method of renormalization in quantum field theory. The method was presented at my guide's festschrift, and published in the proceedings,[12] but never advertised, or further developed. Hence, it is hardly known, and the following is a brief explanation.

3.2 The renormalization problem

The propagators of quantum electrodynamics are what mathematicians call fundamental solutions of PDEs. The fundamental solution of the Dirac equation is the spinor propagator, while that of the relativistic wave equation or Klein-Gordon equation is the photon

propagator. These are also known as Green functions.

These propagators involve entities like the Dirac δ function, which are regarded as generalised functions or Schwartz distributions. On the orthodox formalist exposition of the Indian calculus, as found in university calculus texts today, the derivative is defined as a limit. This definition forces a differentiable function to be continuous. However, in a situation like that of shock waves, the need arises in physics to differentiate a discontinuous function.

The Schwartz theory of distributions modifies the conventional calculus of limits, by allowing discontinuous functions (like the Heaviside jump function) to be infinitely differentiated. However, the limitation of the new theory is that generalised functions or distributions cannot be multiplied: the Schwartz theory assigns a meaning to δ, but not to the product $\delta \cdot \delta$.

Some writers on shock waves, such as Taub,[13] have wrongly maintained (on "physical grounds") that this is a trivial problem . Taub wrote "Fortunately the product of such distributions (as arise) is quite tractable." He argues as follows. Let θ denote the Heaviside function,

$$\theta(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0, \end{cases} \qquad (18)$$

(the value at 0 does not matter[2]). Then, $\theta^2 = \theta$, so that differentiation gives $2\theta \cdot \dot{\theta} = \delta$. But $\dot{\theta} = \delta$ hence $\theta \cdot \delta = \frac{1}{2}\delta$. The problem with this is that we also have $\theta^3 = \theta$, so that $3\theta^2\delta = \delta$, so that, since $\theta^2 = \theta$, $\theta \cdot \delta = \frac{1}{3}\delta$. Another example is that of $(x^{-1} \cdot x) \cdot \delta = \delta \neq 0 = x^{-1} \cdot (x \cdot \delta)$. Thus, neither the product rule for differentiation nor the associative law may be safely assumed in dealing with products of Schwartz distributions.

The infinities of quantum field theory have long been believed to arise because (Fourier transforms of) products of propagators (generalised functions) enter into the S-matrix expansion. Thus, for example, if $\hat{}$ denotes the Fourier transform, $(\delta \cdot \delta)\hat{} = \hat{\delta} \otimes \hat{\delta} = 1 \otimes 1 = \int_{-\infty}^{\infty} 1 dx$, where $\otimes$ denotes convolution, and it is blindly assumed (as in quantum field theory) that a Fourier transform carries products to convolutions (even when the former is undefined!).

What I showed long ago was that this belief is wrong: the problem does *not* lie with products of distributions alone. I defined a natural product of distributions[14], still the only such definition which works for both classical physics and quantum field theory.[15] This definition earlier used non-standard analysis, but the definition actually works perfectly well with a so-called non-Archimedean ordered field, such as the number system of "unexpressed fractions" (rational functions) routinely used from the 5th c. by traditional Indian mathematicians while developing the calculus. Anyway, with my definition, all propagator products arising in quantum field theory are finite, in one dimension.[16] My analysis[12] identified the problem as really that of defining *compositions* not products.

[2]since the Schwartz theory is based on the Lebesgue integral where the value of a function at one point is irrelevant, since a point has measure zero.

What difference does that make? The difficulties which arise with compositions are different from those that arise with products. Thus, I did define compositions along with products,[14] but the issue is as follows. The propagators need to be defined on the null cone $\lambda = 0$. How should we define the composition $\delta(\lambda)$? For any hypersurface $\Sigma = 0$, we can naturally define $\delta(\Sigma)$ just as $\delta(n)$, locally, wherever there is a unique normal to Σ and n denotes the coordinate normal to Σ in Gaussian normal coordinates, so that the equation of Σ locally is $n = 0$. That is we can define $\delta(\lambda)$ everywhere on the null cone except at its vertex. For the particular case of the δ function, we can extend the definition of $\delta(\lambda)$ even to the vertex of the null cone. But in the case of a general distribution f, there is a geometrical difficulty in defining $f(\lambda)$ at the vertex of the null cone because there is no unique normal vector there.

This understanding immediately suggests a very simple and elegant solution to the problem of the infinities of quantum field theory.[12] Namely, eliminate that vertex and replace the support of the propagators by a Lorentz-invariant hyperboloid. Unlike a cutoff, this preserves the Lorentz invariance of the theory, which is *essential* for all current physics as already noted. Unlike a regularisation left on, the support of the propagators is not fuzzy, so interactions do not creep outside the null cone, and positivity of energy is preserved.

Changing the propagators is equivalent to changing the underlying PDEs (Dirac equation, Klein-Gordon equation), of which these propagators are fundamental solutions. That does not really matter, since all calculations are actually done only with the propagators. For example, look at the way we use the Green function to get solutions of Maxwell's equations. Once we have the propagator, or the Green function, we also have the solution, and we don't really need to refer back to the equation.

3.3 Back to classical electrodynamics

The point of this long digression into quantum electrodynamics is this. Can this solution to the problem of infinities in quantum electrodynamics be applied to get rid of the infinities of classical electrodynamics? Indeed it can! This was done some time ago.[17] This is described below using the covariant formulation of electrodynamics to emphasize that everything is done in a Lorentz covariant way.

As regards the Lorentz-invariant hyperboloid, which replaces the null cone, there are two possibilities. A hyperboloid of one sheet would give a Lorentz-covariant model of a spatially extended particle, but this does not give radiation reaction. To get radiation reaction, we need a hyperboloid of two sheets, or what one might call "particles extended in time". We denote the separation by d, assumed to be a constant for the moment.

4 Modified Maxwell equations

4.1 Obtaining the new equations of motion

Thus, the new retarded Green function for classical electrodynamics is given by

$$G_r(\mathbf{x},\mathbf{y}) = \delta((\mathbf{x}-\mathbf{y})^2 + d^2)\theta(x^0 - y^0), \quad (19)$$

where θ is, as before, the Heaviside step function, and δ is its derivative the Dirac delta. Exactly how this changes Maxwell's equations has been worked out, but is irrelevant as explained above.

We follow the original article,[17] and use the metric diag$(-c^2, 1, 1, 1)$, i.e.,

$$||\mathbf{x}||^2 = x^\mu x_\mu = -c^2(x^0)^2 + \sum_i (x^i)^2. \quad (20)$$

For vectors satisfying $(\mathbf{x}-\mathbf{y})^2 = -d^2$, a Lorentz transformation cannot change the sign of $x^0 - y^0$. Hence, the Green function in (19) is Lorentz invariant.

Scalar and vector potentials are obtained as usual.

$$A_\mu(\mathbf{x}) = \frac{1}{2\pi\epsilon_0 c}\int j_\mu(\mathbf{y})G(\mathbf{x},\mathbf{y})d^4\mathbf{y} + \partial_\mu\chi, \quad (21)$$

where χ is an arbitrary scalar function which vanishes in the Lorenz gauge.

For a point charge q, with worldline $\alpha^\mu(s)$ and proper time, s,

$$F_{\mu\nu}(\mathbf{x}) = \frac{q}{4\pi\epsilon_0 c\,(\zeta\cdot\dot\alpha)^2}\left(\ddot\alpha_{[\mu}\zeta_{\nu]} - \frac{\dot\alpha_{[\mu}\zeta_{\nu]}\,(c^2 + \zeta\cdot\ddot\alpha)}{\zeta\cdot\dot\alpha}\right). \quad (22)$$

Dots now denote derivatives with respect to proper time, evaluated at retarded/advanced time, τ_r, τ_a, obtained as the solution of

$$||\mathbf{x}^\mu - \alpha^\mu(\tau)||^2 + d^2 = 0. \quad (23)$$

The retarded time τ_r is the solution for which $\mathbf{x}^0 > \alpha^0(\tau_r)$, while the advanced time τ_a is the solution satisfying the opposite inequality. Further, the vector ζ is defined as the retardation vector pointing from the retarded position to the current position: $\zeta^\mu = \mathbf{x}^\mu - \alpha^\mu(\tau_r)$, and similarly in the advanced case, using the advanced time τ_a instead. For a slow moving particle, the delay $\tau_d \equiv \tau - \tau_r \sim \frac{d}{c}$.

The equation of motion of a charged particle obeying the modified Maxwell equations is

$$\ddot\alpha^\mu = \frac{q}{m}\dot\alpha_\nu F^{\mu\nu}, \quad (24)$$

where $F^{\mu\nu}$ is the net field strength and includes the field from the particle.

This equation looks the same as in the Maxwellian theory, but because of the separation constant d there is retardation involved even in the case of self-action, where $F^{\mu\nu}$ is solely the self-field. Hence, the resulting equations of motion for even a *single* accelerating charged particle is now an FDE, *not* a third-order ODE.

4.2 Consequences of changing Maxwell's equations

What difference does that make? The difference is this. Unlike the runaway solutions of the 3rd-order ODE, arising from Maxwell's equations, this FDE arising from

the modified Maxwell's equations has globally bounded solutions. However, locally the value of the radiation reaction remains roughly the same as that described by the third order term. This is as expected, since the modification of Maxwell's equations is "small", since the delay involved, $\tau_d \sim \frac{d}{c}$, is very small, and roughly the same as the relaxation time of the electron defined in (17), if we suppose that $d \sim r_e$.

Most importantly, we reiterate that this gives us a way to actually solve the problem of the motion of a classical charged particle with radiation reaction. Thus, unlike all the previous attempts in the past century, this modification of Maxwell's equations resolves the problem of how to actually calculate the motion of a charged particle with radiative damping.

There are no doubt technical difficulties in obtaining a numerical solution. For example, for the case of the hydrogen atom, the problem is numerically stiff: there are two widely different time-scales in the problem: the time scale of the radiation reaction and the time scale of orbital motion. Nevertheless, there does exist a code called RADAR to solve numerically stiff FDEs,[18] and we have actually used it. The details are in the original paper.[17]

However, it should be clear by now that the rigorous solution of the classical electrodynamic 2-body problem, and even the 1-body problem, is a complex matter. But, why should we bother to find a solution to such FDEs? Don't we already know that quantum mechanics is the right theory? Is it worth the effort? Is any of this going to lead to quantum mechanics?

We will see the answers to these questions in the next part of this series of articles.

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NEWS: Rahul Basu Memorial Award 2014

for Best Thesis in any area of High Energy Physics.

Nominations invited

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http://www.imsc.res.in/rahul_basu_memorial_award_2014

About the award:

This award is instituted in memory of the late Prof. Rahul Basu, one of India's leading High Energy physicists and a warm and enthusiastic member of the HEP community whose presence is greatly missed. The award, coordinated through the Indian Physics Association, will be given for the two most outstanding Ph.D. theses from India, across all areas of High Energy Physics (theoretical and experimental), during a 2-year period starting from September 1 of each even year. To be eligible for the 2014 award, the thesis should have been accepted with a Provisional Degree Certificate from a recognised university, dated between **September 1, 2012 and August 31, 2014**.

The winners, to be selected by an expert committee, will receive a cash prize (currently Rs 10,000) and a citation. The awards will be presented at the DAE symposium in High Energy Physics. The 2014 award (the second one in the series) will be given at the DAE symposium to be held at IIT-Guwahati from December 8-12, 2014.

Nomination procedure:

Nominations for this award are invited from faculty members working in any aspect of High Energy Physics at a university or research institute. They should write to **rahul.basu.award@gmail.com** providing the name and email address of the nominee, and a brief (one paragraph) description of the nominee's work. The name of the nominee should also appear in the Subject line of this email.

Thereafter the award committee will contact the nominee to submit his/her thesis, along with a detailed CV including list of publications and a copy of the Provisional Degree Certificate. The nominee will be asked to request two referees (one of whom can be the nominator) to write directly to the same email address in support of the nomination. In all communications, the nominee's name should appear in the Subject line.

Winners will be selected by an expert committee whose decision will be final

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Deadline:

The last date to receive nominations will be **October 15, 2014**. Short-listed nominees may be contacted later on by the expert committee and asked to provide additional details as required.

Award ceremony:

The winners will be invited to receive the award at a ceremony during the DAE Symposium in HEP, to be held at the Indian Institute of Technology, Guwahati from December 8-12, 2014 and to deliver a seminar on their thesis work at the award ceremony. Those who cannot attend in person may be allowed to deliver the talk over the internet.