ISSN 0970-5953

Vol. 38 No.3

October-December 2024

Physics Education

Quarterly e-Journal Devoted to Physics Pedagogy

Sir C.V. Raman's Study of Drums using Mridangam

www.physedn.in www.indapt.org.in

Volume 38, Number 3

In this Issue:

Editorial

Revitalizing Physics Education:
A New Chapter

As we proudly present the third issue of our Physics Education Journal, we mark a significant milestone in our journey to revitalize physics education. After a dormancy of almost three years, our first and second issues were met with enthusiasm and appreciation from the physics community. This renewed energy and commitment to quality publishing have set the stage for our continued growth and relevance.

A Commitment to Excellence

Our editorial team has worked tirelessly to ensure that each article meets the highest standards of academic rigor and relevance. This issue promises to deliver insightful research, innovative teaching methods, and thought-provoking discussions that will enrich the physics education landscape.

In This Issue

This third issue features a diverse range of articles, including innovative laboratory experiments on drums by Sir C.V. Raman, a discussion on tensors, a new approach to polynomial algebra and a simple demonstartion of amplitude modulation. These contributions reflect our commitment to addressing the needs and interests of physics educators and learners.

Gratitude and Acknowledgment

We extend our heartfelt gratitude to our authors, reviewers, and readers for their unwavering support and contributions. Your dedication to physics education is the driving force behind our journal's resurgence.

Looking Ahead As we move forward, we remain committed to publishing high-quality research and fostering a community of physics educators and learners. We invite you to engage with our content, share your thoughts, and contribute to our future issues.

Professor O.S.K.S Sastri
Editor-in-Chief
Physics Education Journal (IAPT)

Introducing Raman's study of drums in laboratories

Nishanth.[1] and Udayanandan K. M.[2]

[1]School of Pure and Applied Physics, Kannur University,
Payyanur Campus, Payyanur, Kerala - 670 327, India.
mailnishanthp@yahoo.com

[2] Department of Physics, S. N. College,
Vadakara, Kerala - 673 104, India.
udayanandan@gmail.com

Submitted on 19-10-2021

Abstract

In many college laboratories across India, students conduct or replicate experiments performed by scientists worldwide. However, the groundbreaking experimental work of Sir C. V. Raman, one of India's most brilliant experimentalists, is often overlooked. To address this gap, we propose a simple and cost-effective method to introduce Raman's classic experiment on drums into college laboratories.

1 Introduction

Sir C. V. Raman's study about drums was first published in 1920, while he was at Kolkata [1]. In 1922 he gave a detailed account on the Indian contribution of making musical drums [2]. After 15 years of detailed studies, while in Bangalore, he published a breakthrough paper on the harmonics generated by Mridangam [3], which is actually the beginning of a systematic study of musical drums. Out of many musical drums in India, Mridangam is the most ancient instrument and it is played in concerts and art forms. Many theoretical and experimental studies about the vibrations of the drum head of Mridangam were published later [4–7]. Our paper gives a simple method for the study of harmonics in musical drums in an undergraduate laboratory very easily and cost-effectively. Before going to our method a brief idea about the contributions of Raman in the field of drums is given in the next section.

2 Raman's contribution

Raman, while working at Kolkata in Indian Financial Service, during his spare time, studied the acoustics of Indian drums [8].

The most important drum that attracted Raman with its highly harmonious tones was Mridangam. One of the objectives of Raman on studying was to find the differences in the construction of the Indian musical instruments from other regions of the world and to find the most important component that contributes to the tonal qualities of Indian instruments. Raman found that the

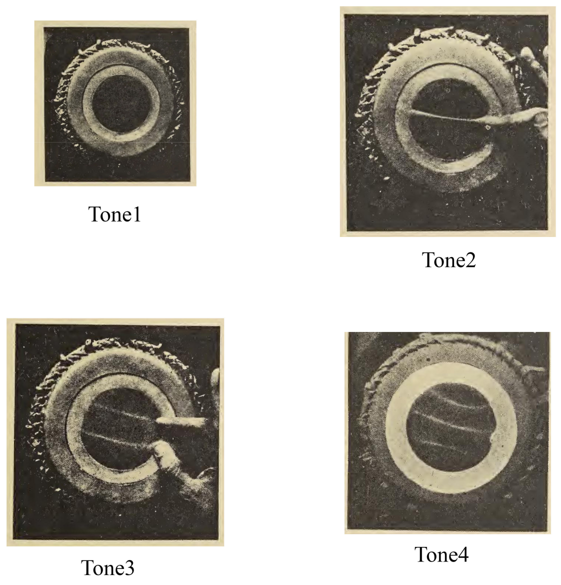

Figure 1: The production of tones on Mridangam head as in Reference [9]

black paste on the drum head is the cause of shifting the non-harmonic overtones in a circular membrane to harmonic tones in Indian instruments like Mridangam. To excite different harmonic tones, Raman sprinkled lycopodium power on the drum head and the drum was played with experienced artists [10]. By hearing and seeing the tones produced, Raman identified the harmonics as there were no modern techniques for study at that time. The images of Mridangam drum head excited by Raman are given in Figure 1. The frequency ratios obtained by Raman [3] are given in Table 1

Mode	Ratio
0,1	1.000
1,1	2.00
2,1	3.00
0,2	3.25
3,1	4.00
1,2	4.321
4,1	5.00
2,2	5.34
0,3	5.167

Table 1

This is a historical discovery that was published in Nature [1] a prestigious journal of physics. Before going to the details of our study we will have an understanding of harmonics and their relevance in music. What is meant by a mode will be discussed in another section.

3 Harmonics

A sound produced by any instrument consists of a collection of frequencies. The lowest frequency among them is termed as the fundamental frequency and higher frequencies are termed as overtones. These overtone frequencies and the fundamental are called together as harmonics when they have an integer multiple number relation. Consider the set of frequencies in Table 2.

Frequency(Hz)	Ratio
150	1
210	1.4
260	1.73
300	2
330	2.2
450	3
600	4

Table 2

Here the lowest frequency is 150Hz and hence it is the fundamental frequency and all other frequencies are called overtones of fundamental. But only 300Hz, 450Hz, 600Hz are harmonics along with fundamental since they have integer ratios. Hence the 150 Hz is called first harmonic, 210Hz and 260 Hz are called first and second overtones but not harmonics as they do not form integer ratio with 150 Hz. The 300Hz is called third overtone and second harmonic and so on. The central loaded region of the Mridangam is called the **Karane**(Figure 2).

Figure 2: The black loading on Mridangam head

4 Modes of vibration of a simple circular membrane

In India many drums like Dhol, Timila etc use one or more layers of animal skin stretched over a wooden shell and are played with either hand or stick. The vibrations of a single layer of animal skin attached to the drum head are studied as a circular membrane with a fixed boundary. A mode is a pattern of vibration in which the whole membrane vibrates except certain points that remain at rest called nodes. The two-dimensional wave equation represents the vibration of a circular membrane [11] is.

$$\nabla^2\psi(r,\theta) = \frac{1}{c^2}\frac{\partial^2\psi(r,\theta)}{\partial t^2}$$

Here $\psi(r,\theta)$ is the transverse displacement, r is radial component of displacement and θ is the angular component of displacement and $c = \sqrt{\frac{T}{d}}$ where T is the tension produced on the membrane and d is the mass density. The r-dependent solution for the membrane is [12]

$$\psi(r) = AJ_n(kr) + BY_n(kr)$$

where $J_n(kr)$ is the Bessel function of first kind with order n and $Y_n(kr)$ is the Bessel function of second kind with order n. At the center of the membrane $r = 0$. The Bessel function of second kind has no finite solution at the centre. Hence we consider only first part of the solution as

$$y(r) = AJ_n(kr)$$

At boundary $r = a$, since it is fixed we get

$$J_n(ka) = 0$$

There are numerous solutions for this equation given by x_{nm}. These solutions are termed as zeros of the Bessel function or roots of the Bessel function. The number of the roots is represented by m and the order of the Bessel function is represented by n. Thus we get

$$ka = x_{nm}$$

Physically on a drum head, the non-vibrating points originate along the diameter or along the circumference of a circle called nodes. The straight line and the circle created by zero vibration points are then called nodal diameter and nodal circle respectively. The number of nodal diameters

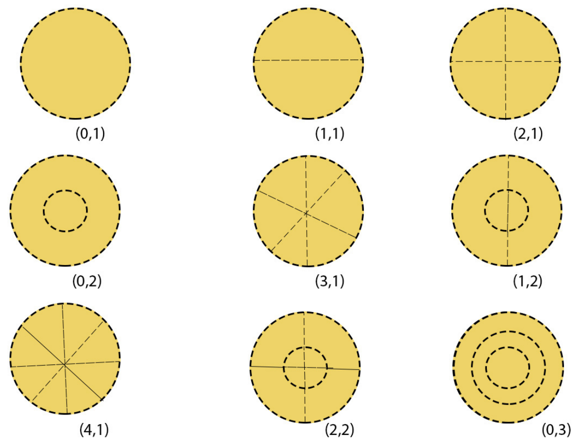

Figure 3: The modes of vibration of circular membrane

formed on a drum head is represented by the order of the Bessel function n and the number of nodal circles formed on the drum head are indicated by the number of roots of the Bessel function m. The first few modes of vibration in circular membrane are shown in Figures 3 and 4. The frequency ratios

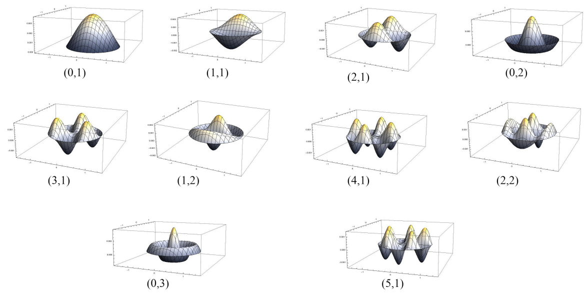

Figure 4: The modes of vibration of circular membrane

of first few modes of vibration of a circular membrane calculated using the zeros of the Bessel function [13] are given in the Table 3

Mode	Ratio
0,1	1.00000
1,1	1.59335
2,1	2.13556
0,2	2.29545
3,1	2.65311
1,2	2.91733
4,1	3.15548
2,2	3.50016
0,3	3.59851
5,1	3.64749

Table 3

The lowest mode is the fundamental mode which has only one nodal circle and no nodal diameter. The axis-symmetric vibration of modes do not generate nodal diameters and hence all modes with $n = 0$ have only nodal circles. The first nodal diameter is formed by the vibration of the second mode that has one nodal circle also. This mode generates the first overtone frequency which is 1.5933 times the fundamental. Out of nine overtone generating modes given in the table, second, third, fifth, seventh and tenth modes have one nodal circle and 1, 2, 3, 4 and 5 nodal lines respectively. Two nodal circles are formed in the fourth, sixth and eighth modes with 0,1, and 2 nodal lines respectively. The changes in the vibration in angular direction generates these nodal diameters. The maximum number of nodal circles are formed by the ninth mode with no nodal diameter. From Table 3, it can be seen that all the overtones have a non-integer ratio with fundamental and due to this none of them form harmonics. Thus, the sound produced by an ordinary circular membrane drum head is not musical. Now let us study the music produced by Mridangam in the next section.

5 Music produced by Mridangam

Mridangam is a drum built with wood and air is enclosed inside the instrument after construction [14]. A Mridangam is shown in Figure 5. A black region with high density is made on the drum head constructed with goat skins by rubbing fine paste made of rice, and stone that is known 'Puranakeedam'. The cow or buffalo hide is used to make an annular region on the drum head and ropes are used to stretch the drum head tightly on the rim. The circular gap between the loaded region and annular region is filled with materials like tiny plastic balls, dried stems of wheat or paddy. Harmonic tones are produced by five strokes on the right loaded head in Mridangam. The strokes include Dheem, Arachappu, Naam, Chappu and Dhin [15]. The stroke Dheem is produced by striking the loaded region on the drum head with fingers in the right hand and removing immediately. The Arachappu is produced by the little finger, strikes along the diameter in the loaded region away from the centre and other fingers are used as support. Naam stroke is made by striking at the edge of the annular membrane with for finger and keeping the dark region at rest by placing the ring finger at its circumference. The Chappu is elicited by striking with the little finger at the circumference of the loaded region towards the centre. The final stroke Dhin is produced by striking fore finger at slightly inward to the circumference of the loaded region and the ring finger is placed on the circumference at 60-degree distance.

Figure 5: The Mridangam

6 Our method

The audio samples of Mridangam strokes in MP3 format with a duration of few seconds are collected. The samples are placed in a folder on a mobile phone. For analysis, the **Visual analyzer** software freely available on the internet is used. The installation is done by double-clicking on the executable file and following the instructions. Open the software using the icon on the desktop. A screenshot of the software window is shown in Figure 6. Connect headset or mike on the

Figure 6: The screenshot of Visual Analyzer software

laptop, place the mobile phone near the talking point of the headset. Click on the **ON** button in the menu bar before the settings menu and play the sound from the mobile phone. The capture spectrum button on the panel in the left bottom end in the software window. A new window showing the frequency spectrum will appear. Left-click and drag to zoom the spectrum at the beginning of the axis. Place the mouse at each peak in the frequency spectrum and corresponding x-axis and y-axis values are seen at the bottom. The spectrum is saved as a PDF file using

$$File \longrightarrow Printspectrum$$

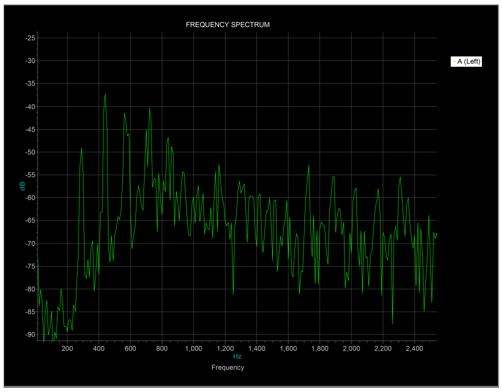

Figure 7: The frequency spectrum of Naam stroke

7 Results

The Dheem stroke excites the lowest mode or fundamental mode. For all other strokes, the fundamental is slightly higher than one. This indicates that the other strokes do not vibrate in fundamental mode since the fingers placed on the drum head suppress it. In such strokes, the higher harmonics that are in the integer relation are identified by the brain and generates a pitch sense around the frequency of the second harmonic present in the stroke. To tune the drum head, artists use Arachappu or Naam strokes. Hence we plot Naam stroke as an example that is given in Figure 7. Here we used a Mridangam tuned to pitch D3 and its standard frequency is 146.83Hz. The frequency of the peaks present in the spectrum is given in Table 4.

Number of peak	Frequency (Hz)	Ratio
1	160.059	1.10
2	290.61	2.00
3	439.169	3.02
4	560.717	3.85
5	720.53	4.95

Table 4

8 Discussion

Here, the third peak is the most prominent, indicating that the third harmonic tone is heard when the Naam stroke is excited. Similarly, the most prominent peak in the spectrum is used to determine the order of the harmonic tone produced. In other words, the modes vibrating with maximum energy are identified from the dominant peak.

C. V. Raman extensively studied the acoustics of the Indian drum Mridangam and published his findings after years of meticulous evaluation. Today, the same tonal characteristics can be analyzed with significantly less effort and time using modern software techniques. The study of musical drums and their harmonics can serve as an engaging experiment for undergraduate students, helping them understand and appreciate C. V. Raman's contributions to the field of acoustics.

9 Conclusions

We believe that the best way to honor Raman, India's greatest scientist, is by studying and replicating his experiments. His research is so insightful that dedicating a section of college laboratories to exploring his contributions would be highly beneficial. In this paper, we demonstrate that Raman's study of harmonics can be easily conducted using a computer, a resource readily available in most college laboratories.

10 Acknowledgments

The authors wish to express their gratitude to Mr. Arjun, a skilled Mridangam player, for his invaluable assistance in identifying, distinguishing, and recording the various tones produced by the instrument.

References

[1] C. V. Raman and S. Kumar (1920). Nature, 104, 500.

[2] C. V. Raman (1922). Asutosh Mookerjee Silver Jubilee Volume 2, 179.

[3] C. V. Raman (1934). Proc. Indian Acad. Sci. (Math. Sci.) 1, 179.

[4] R. B. Bhat (1991). J. Acoust. Soc. Am. 90, 1469.

[5] V. Dubey and I. P. Krishna (2021). Appl. Acoust. 181, 108121.

[6] R. Krishnamurthy, I. Hempe, and J. Cottingham (2008). J. Acoust. Soc. Am. 123, 3606.

[7] B. S. Ramakrishna, M. M. Sondhi (1954). J. Acoust. Soc. Am. 26, 523.

[8] S. Banerjee (2014). Phys. Perspect. 16, 146.

[9] C. V. Raman and Sivaraj Ramaseshan (editor). Scientific Papers of C. V. Raman, Volume 2, (Indian Academy of Sciences, 1988) http://archive.org/details/scientificpapers0000rama

[10] A. Jayaraman. C V Raman – A Memoir, (Indian Academy of Sciences, 2017)

[11] A. W. Leissa. Membranes. In Encyclopedia of Vibration (pp. 762–770), (Elsevier,2001)

[12] D. N. S. Handayani, Y. Pramudya, S. Suparwoto and M. Muchlas (2018). J. Phys. Theor. Appl. 2, 83.

[13] K. T. Tang. (2007). Mathematical methods for engineers and scientists (Vol. 3), (Springer, 2007)

[14] V. U. Sivaraman, T. Ramasami and M. D. Naresh (2010). Science for Musical Excellence (Sixth Raja Ramanna memorial Lecture).

[15] K. Varadarangan Acoustic properties of the South Indian Mrudanga. In: https://karunyamusicals.com/knowledge-centre/

Tensors: Significance and Features

S.S.Z. Ashraf[1]

[1]Department of Physics, Aligarh Muslim University, Aligarh 202002, India.
ssz_ashraf@rediffmail.com

Submitted on 11-12-2021

Abstract

At the first face-off with Tensors, they appear to be scary and formidable. Even Einstein had to struggle a lot to master it. But once well versed with the subject Einstein used it aptly in the formulation of his masterpiece General Theory of Relativity (GTR), whose fame in due time made the subject of tensors synonymous with GTR in Physics. Earlier, however, with their little use in other areas of physics, students not opting for GTR could turn a blind eye to them. But since GTR much water has flown down the tensor pipeline, and the subject has evolved a lot with numerous applications not only in Physics and Mathematics but in fields as varied as Computer Science, Chemistry, Geology, Statistics, Medicine, Engineering, etc. In Physics, it is now regarded as an indispensable tool for the description of all the four fundamental interactions. Further, an operation on tensors called tensor product is a pre-requisite for the description of quantum states when two or more quantum systems get together, and also, the entangled states in their joint vector space need tensors for their expression. Very significantly, the fifth aspect of tensors, apart from its ability to represent invariance, anisotropicity, many quantum systems states and entanglement, is the capacity for large data storage, which is an artifact of the fact that high-rank tensors can be effectively represented by multi-dimensional hyper matrices. This feature of tensors has come to great advantage in Computer Science, where it is utilised for organising or storing large data and data mining with bearing on machine learning, deep learning, tensor imaging, face recognition, computer vision, etc. This article is a modest attempt (as the subject is deep and profound and cannot be justified in an article of over a dozen pages) to make accessible the features of tensors and their significance to the undergraduate students. The goal here is not to provide the students a working knowledge of tensors but to entice them by showing them the wonderful world of tensors so that they learn it on their own.

1 Tensors: Etymology, Origin and Development

It appears that the English word *Tensor*, which owes its origin to the Latin word *Tensus* meaning **Tension**, has an influence on its import. *Box 1* quotes a few tensor anecdotes that testify to tensor's reputation or notoriety for being daunting. Incidentally, the first tensor used in physics has a close connotation with its meaning, as it is none other than the famous *stress* tensor. A tensorial expression, represented by some symbols adorned with multiple indices in subscript and superscript fashion, projects a frightening sight to anyone who wants to comprehend its meaning. In case the symbol appears pleasing to some gutsy person and emboldens him/her to read the modern highly abstract definition *'A tensor is a binary covariant functor [1] that represents a solution for a co-universal mapping problem on the category [2] of vector spaces over a field,'* will certainly spin his/her head. This no doubt looks quite intractable at the first sight. But a little familiarity with tensors makes one regardful of how important a tool it is to express equations of physics, notwithstanding the other important applications which the tensors lend themselves to.

The word *'tensor'* was introduced by William Rowan Hamilton (1805–1865), initially to describe something different from what is now meant by a tensor (namely, the norm operation in a certain type of algebraic system now known as *Clifford algebra*). The contemporary usage was introduced by Woldemar Voigt around 1898. The concept of tensors has its origin in the development of differential geometry by mathematical stalwarts no less than Carl Friedrich Gauss (1777-1855) and Bernhard Riemann (1826-1866). Later, Elwin Bruno Christoffel's (1829-1900) work in differential geometry, particularly the connection formulae obtained by him to express covariant derivatives, paved the way for tensor calculus. Gregorio Ricci-Curbastro (1853-1925) and his Student Tullio Levi-Civita (1873–1941) generalized Christoffel's ideas and developed them further to institute the concept of tensors and absolute differential calculus. *Figure 1* contains photographs with names printed below each of a few of the progenitors of the subject of tensors. The absolute differential calculus, later known as tensor calculus, forms the mathematical basis of the general theory of relativity, which popularized the subject by leaps and bounds. From 1920 onwards, tensor concepts pro-

Figure 1: Few Tensor Progenitors Photographs with their names.

gressed to newer, more abstract areas that is from differential geometry to topological algebra, Topological algebra [3] and more recently, to category theory. It won't be an exaggeration to say that the study of tensors is a study in the progress of mathematical thought. The stated tensor definitions in *Box 2* allude to this evolution of mathematical ideas.

- General relativity is formulated completely in the language of tensors. Einstein had learned about them with great difficulty from the geometer Marcel Grossman.
- Levi-Civita during 1915-17 initiated a correspondence with Einstein to correct mistakes Einstein had made in his use of tensor analysis.
- Albert Einstein in a letter to Tullio Levi-Civita wrote: I admire the elegance of your method of computation: it must be nice to ride through these fields upon the horse of true mathematics while the like of us have to make our way laboriously on foot.
- When an interviewer questioned, Professor Eddington, is it true that only three people in the world understand Einstein's theory? Eddington retorted: Who is the third?
- A professor at Stanford once said, 'If you really want to impress your friend and confound your enemies, you can invoke tensor products... people run in terror from the ⊗ symbol.

BOX 1. Tensor Anecdotes.

2 Tensor Applications

These days abundance of literature on tensors being copiously produced by mathematicians, physicists, computer scientists, statisticians and engineers as well as experts in other scientific fields signify to the importance that tensors hold in Science and Engineering. As mentioned above, few decades before tensors were almost synonymous with General Relativity- except for a minor use in all other branches of Physics. The realization that gauge fields are geometrical objects has made the geometrical (coordinate-independent) aspect of tensors become more and more significant in the study of all interactions as all fundamental interactions including gravity are deemed to be different manifestations of the same super force.

In recent decades, relativistic quantum field theories, gauge field theories, and various unified field theories have all used tensor algebra analysis exhaustively. Also tensor products naturally arise in quantum mechanics as a description of many particle state space because they can take into account the superposition aspect of quantum states when separate quantum systems are brought together. Further the fast burgeoning field of quantum computation hinges on the concept of entangled states which need tensors for their formulation. In mathematics tensors are used in Differential Geometry, Differential Equations, Spectral Theory, Continuum Mechanics, Fluid Dynamics, Multilinear systems in Numerical Algebra, Tensor complementarity problems, Optimization, etc.

One of the most important applications of tensors is to tensor decomposition that is presently used for applications in numerous varying fields. Though tensor decomposition methods have appeared as early as 1927, but they remained unused in computer science field as late as the end of 20th century. An early use of tensor decomposition was sought in the area of psychometrics which deals with intelligence evaluation and other personality characteristics. But in the last two decades, a growing computing capacity and an increasing familiarity with multilinear algebra have led tensors to emerge in a big way in the earlier untouched areas of statistics, data science and machine learning. In data science, real data are often in high dimensions with multiple aspects and tensors provide elegant theory and algorithms for web data mining, face recognition softwares, higher order diffusion tensor imaging in medical imaging, psychometrics, chemometrics, neuroscience, graph analysis, fluorescence spectroscopy, geophysics, etc. In each case, data is compiled into a multi-way array or a hyper matrix and the essential features of the data are isolated by decomposing the corresponding tensor into sum of rank one tensors.

A. Poorman's Definition; Rather Impression
- Tensor is nothing but index gymnastics played with certain rules.

B. Heuristic Definition
- Tensor is what that transforms like a tensor.

C. Canonical Definitions
Tensor(s) is(are);
- a generalisation of vectors & co-vectors.
- an invariant abstract or geometric object with a magnitude and several directions.
- a mathematical entity that transforms according to certain transformation laws.
- a multi-dimensional array of numbers or a n-dimensional generalization of a matrix.
- are just vectors in a special vector space or an element of tensor product space.
- a multi-linear operator that maps vectors and co-vectors to real numbers.
- a binary covariant functor.

BOX 2. Qualitative definitions of Tensors.

3 Approaching Tensors

In physics any quantity that has both magnitude and direction is a vector. Displacement, velocity, acceleration, and force are few examples of mechanical vectors. In three dimensional Cartesian space, a vector is represented by its x, y, z components. If we multiply this vector by a scalar quantity, all the three components of the vector scale up proportionately or, in other words, the vector changes its magnitude without changing its direction.

What if we want to create a new vector with a different magnitude as well as direction than the initial vector? Multiplication by a scalar only changes the magnitude. Taking the inner product with another vector turns it into a scalar, and in this way, the direction too is lost. Forming the cross product with another vector, though it changes the direction, always does so in the normal direction. So, for changing direction in an arbitrary way, we either take the *outer product* of a vector with another vector and obtain a second-rank tensor having a magnitude and two directions, or multiply the initial vector by a new mathematical entity called a *tensor* and obtain a tensor of higher rank, having a magnitude and multiple directions. *Table 1* presents the resultant quantities obtained from various multiplicative products of scalars and vectors, along with their examples in physics in three-dimensional Cartesian space.

A physical example of a tensor of rank two is force acting on a plane surface area. In this case, both the magnitude and direction of the force, and the size and orientation of the area, will determine the total effect. The size of the area and its orientation can be represented uniquely by a vector whose magnitude is proportional to the area size and whose direction is normal to the surface. Therefore, the effect of the force upon the surface depends on two vectors, the force vector and the area vector, and hence is described by a tensor of second rank. Second-rank tensors appear in physics when physical quantities exhibit anisotropic behaviour in the system, often in a "stimulus-response" mode, as discussed in the next section. In general, a second-order tensor, which takes in a vector of some magnitude and direction, returns another vector of a different magnitude and direction. If we take into consideration the components of force, each of the components acting on each component of the area vector, then there are nine terms altogether, which can succinctly be arranged in matrix of order 3 representing the total stress. So tensors can thus be represented by arrays, and manipulated in a manner reminiscent of matrix manipulation. The *Figure 2* shows tensors of zero, first, second and third rank as dimensional arrays or matrices. The single and multi-dimensional different stress components and *Figure 3* exhibits the distress tensor of a point in 3D space.

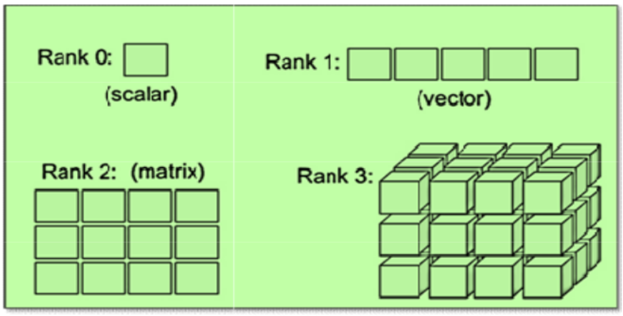

Figure 2: Tensors as multi-dimensional (Hypermatrix) array of numbers.

3.1 Tensor Definition

Tensors have been defined in several equivalent ways. These definitions can be broadly classified into two main types. The first type is traditional and defines tensors using coordinate transformation properties of components of tensors, whereas the second type is more modern and abstract and defines tensors in their component free formulation. We will briefly discuss only the first definition, due to constraints of the article size, but encourage the reader to learn about the second type in the suggested readings.

As remarked, tensors are usually introduced in terms of tensor components trans-

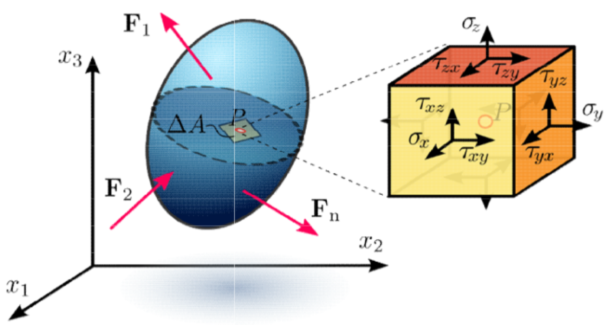

Figure 3: Stress tensor components in 3D space.

formation rule. A tensor consists of tensor components and an underlying basis vectors of the coordinate system in which it is referred to. For the ease of understanding we consider the simplest case of orthonormal Cartesian coordinate system in Euclidean space. In this system the basis unit vectors are constant and so it suffices to give the components of a tensor. Another thing is that the transformations between Euclidean bases are always orthogonal. An orthogonal transformation of tensors from one Euclidean space to another preserves the length, and also makes no distinction between covariant [4] and contravariant [5] tensors. If $T(x) = Mx$ is an orthogonal transformation, we say that M is an orthogonal matrix. And from matrix theory we know that a matrix is orthogonal iff its inverse and transpose are the same, i.e., $M^{-1} = M^{\mathrm{T}}$.

We shall now examine the behaviour of a low order tensor of rank one that is a vector if we move from a two dimensional (2D) Cartesian coordinate system S to another 2D Cartesian system S′. The case of transformation rule of scalars which are tensors of the lowest rank is trivial because scalars are independent of the choice of coordinate system and does not require basis vector for their description. The 2D Cartesian S′ coordinate system in consideration is rotated by an angle ϕ with respect to the S system, as shown in the *Figure 4*. Let $\mathbf{E}$ be an electric field vector lying on a 2D plane, the vector making an angle θ with the x-axis in the S system. Then the components of $\mathbf{E}$ in the S system are $E_x = |\mathbf{E}| \cos\theta$ and $E_y = |\mathbf{E}| \sin\theta$. The coordinates of the electric field vector in the rotated system S′ will be,

$$
\begin{aligned}
E'_x &= |\mathbf{E}| \cos(\theta - \phi) && (1)\\
&= |\mathbf{E}| \cos\theta \cos\phi + |\mathbf{E}| \sin\theta \sin\phi && (2)\\
E'_y &= |\mathbf{E}| \sin(\theta - \phi) && (3)\\
&= |\mathbf{E}| \sin\theta \cos\phi - |\mathbf{E}| \cos\theta \sin\phi && (4)
\end{aligned}
$$

Using $E_x = |\mathbf{E}| \cos\theta$ and $E_y = |\mathbf{E}| \sin\theta$, the above Eqs. (1) and (2) become,

$$
\begin{aligned}
E'_x &= E_x \cos\phi + E_y \sin\phi && (5)\\
E'_y &= E_y \cos\phi - E_x \sin\phi && (6)
\end{aligned}
$$

These transformation equations can be written in matrix form as,

$$
\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}
$$

The matrix form of Eq. (5) can simply be written as $\mathbf{E}' = M\mathbf{E}$, where

$$
M = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}
$$

is a 2×2 matrix and $\mathbf{E}'$ and $\mathbf{E}$ are 2×1 column matrices. Since the elements in the matrix are identified by their row and column positions, the transformation Eqs. (3) and (4) can also be put as

$$
E'_i = \sum_j a_{ij} E_j \qquad (7)
$$

Where the indices i and j take the variables x and y, and the direction cosine coefficients a_{ij} are: $a_{xx} = \cos\phi$, $a_{xy} = \sin\phi$, $a_{yx} =$

Product Name	Initial Entities	Initial Direction	Resultant quantity	Final Directions	Example in 3D Euclidean Space
Scalar	Scalar with Scalar	Zero, Zero	Scalar	Zero	Energy=Boltzmann constant times temperature ($E=K_BT$) (Note in Minkowski 4D space, energy is the Zeroth component of the momentum four vector, so it's not a scalar.)
Scalar	Scalar with Vector	Zero, One	Vector	One	Force= mass times acceleration (F=ma)
Inner	Vector with Vector	One, One	Scalar	Zero	Power=Force times Velocity (P=F.v)
Vector	Vector with Vector	One, One	Vector in Normal direction	One	Angular Momentum*= distance times momentum ($\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$) (*Not in relativity)
Dyad/ Outer/ Tensor	Vector with Vector	One, One	Tensor of second rank	Two	Moment of inertia $\mathbf{I} = \sum m_i\,({r_i}^2\mathbf{1} - \boldsymbol{r_i}\boldsymbol{r_i})$ Where $\mathbf{1} = \boldsymbol{ii} + \boldsymbol{jj} + \boldsymbol{kk}$ is a unit dyadic.

TABLE 1. Resultant tensor quantity from various multiplicative products of scalars and vectors and their examples in Physics.

$-\sin\phi, \quad a_{yy} = \cos\phi.$
Taking partial differential of Eq. (7) with respect to each of the components, and putting them into a matrix yields the following:

$$\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \begin{bmatrix} a_{xx} = \frac{\partial E'_x}{\partial E_x} & a_{xy} = \frac{\partial E'_x}{\partial E_y} \\ a_{yx} = \frac{\partial E'_y}{\partial E_x} & a_{yy} = \frac{\partial E'_y}{\partial E_y} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

Thus, the vector (tensor of rank 1) transformation rule can also be succinctly cast as:

$$E'_i = \sum_j \frac{\partial E'_i}{\partial E_j} E_j \tag{8}$$

Now, by just noting that the transformations in Euclidean space are orthogonal, we can write the inverse transformation equation by inverting the matrix M, which amounts to just transposing the row elements with column elements, that is, replacing a_{ij} with a_{ji}.

$$M^{-1} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

One can easily check that $|M| = |M^{-1}| = 1$ and that $|M||M^{-1}| = I$, meaning that the tensor remains invariant under rotation transformation. Using this fact, the inverse transformation equations $\mathbf{E} = M^{-1}\mathbf{E}'$ can be

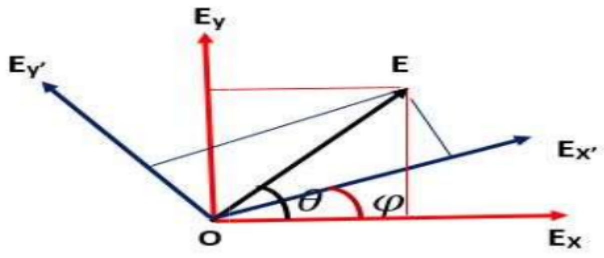

Figure 4: Electric field vector E and their components in 2D Cartesian coordinate system .

written as:

$$E_i = \sum_j a_{ji} E'_j = \sum_j \frac{\partial E_i}{\partial E'_j} E'_j \tag{9}$$

Thus, we infer that a Cartesian vector is an invariant physical quantity that transforms from the S coordinate system to the S' coordinate system according to the Cartesian tensor transformation law given by Eq. (9), and by Eq. (11) for vice-versa. If we generalise this definition, then we have the following definition of a rank n tensor:

A Cartesian tensor of rank n is a set of N^n quantities $T_{ij...m}$, which transform under rotations according to the rule:

$$T_{ij...m} = \sum_p \sum_q \cdots \sum_t T_{pq...t} a_{ip} a_{jq} \ldots a_{mt} \tag{10}$$

where, $a_{ip} a_{jq} \ldots a_{mt}$ are the cosines of the angles between the new and old coordinates.

4 Tensor Features

Though the ability to express invariance[6] is a fundamental property of tensors, besides this main property, four other innate potentialities possessed by tensors come in handy to express various aspects of physical reality in science. These aspects/features or characteristics are namely: Anisotropicity, Many-particle quantum states, Entanglement, and Big data storage capacity.

4.1 Invariance (Covariance of Physical Laws)

The main characteristic of a tensor is that its representations in different coordinate systems depend only on the relative orientations and scales of the coordinate axes at that point, and not on the absolute values of the coordinates. Tensors serve to seclude the intrinsic geometric and physical properties from the coordinate dependent ones. So if two tensors of the same type are equal in one coordinate system, then they are equal in all coordinate systems. Therefore it can be said that the central principle of tensor analysis amounts to the simple fact that tensors remain invariant with coordinate transformations. This implies that equations written in tensor form are valid in any coordinate system as tensor equations look the same in all coordinate systems. This is why the absolute position vector pointing from the origin to a particular object in space is not a tensor because the components of its representation depend on the absolute values of the coordinates.

The physical reality encoded in the laws of physics is universal that is independent of reference frames under appropriate symmetry transformations. So this means it depends on what laws one is talking about, as for instance Newton's laws are invariant with respect to the Galilean transformations and Standard model is invariant with respect to Lorentz transformations. In both these theories there is a preferred set of frames called inertial frames. The theories or the laws are invariant only with respect to which inertial frame one is using. In contrast, general relativity is invariant with respect to general coordinate transformations. And as remarked above, the main characteristics of objects called tensors is that they remain invariant under certain coordinate transformations. So it should be clear that invariance of tensors is subject to transformation rules. One should first talk about the transformations under which one is asking for invariance. Only then, logically speaking, can one talk about tensors. The same object could have different transformation properties with respect to different transformations. For example, Higgs boson (before electroweak symmetry breaking) is an SU(2) doublet while Lorentz scalar. So, as a tensor it will have only one SU(2) index and no Lorentz index.

This entails that if laws of physics are expressed using tensors they become form invariant under appropriate transformations and hence tensors provide the best means to objectively represent the physical reality independent of coordinate systems or observers. In the language of physics if the equations of physics possess the same form in different coordinate systems they are said to be covariant, though the word covariant

Tensors equations are covariant (take the same form) in all coordinate systems

As an illustration of this fact consider a tensor equation of the form as given below:

$$U_j^i = \rho W_j^i \qquad (1)$$

where U_j^i and W_j^i are mixed tensors of rank 2 and ρ is some constant. Under a coordinate transformation U_j^i and W_j^i transform as:

$$U_\beta^\alpha = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial x^j}{\partial y^\beta} U_j^i \qquad (2)$$

$$W_\beta^\alpha = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial x^j}{\partial y^\beta} W_j^i \qquad (3)$$

And the equation (1) transforms as:

$$\frac{\partial y^\alpha}{\partial x^i}\frac{\partial x^j}{\partial y^\beta} U_j^i = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial x^j}{\partial y^\beta} \rho W_j^i \qquad (4)$$

With the help of equations (2) and (3), equation (3) reduces to;

$$U_\beta^\alpha = \rho W_\beta^\alpha \qquad (5)$$

Comparing the old coordinate equation (1) with the new coordinate equation (5) we observe that they wear the same form.

BOX 3. Illustration of the invariance of Tensor equation.

in tensor analysis has different meaning too. The *Box 3* illustrates with an example the covariant nature of a tensor equation. This demand of covariance, that all physical laws should be invariant under transformation between inertial systems led Einstein to the formulation of his theory of special relativity. Also, the need that the Maxwell's equations should be invariant under transformations, and the failure of Galilean transformations to do it, led to the Lorentz transformations. In *Table 2* we present few familiar equations (Newton's 2^{nd} law, Maxwell's equations and Dirac equation) in their usual scalar/vector form and the same in tensorial form.

4.2 Anisotropy (in properties of Material, Fields and Manifolds)

When materials are subjected to some stimulus like mechanical force, electric field, magnetic field, temperature field, etc., they subsequently respond, which respectively may be reflected in some property change such as elongation/deformation, electric current or polarization in dielectrics, magnetization, heat flow, etc. But these responses may also be associated with some other cause/stimulus. For instance, the application of pressure may lead to the polarization of the crystal (Piezo-electric effect) or the influence of a magnetic field may lead to strain in the material (magnetostriction), or the presence of temperature difference can cause electrical potential difference (pyroelectricity). In all such cases, a material/physical property connects the stimulus to the response, like, $P = p \cdot \Delta T, \quad M = \chi \cdot H, \quad \sigma = c \cdot \varepsilon$, etc., where the symbols are defined in *Table 3*. These material properties can be measured in experiments or can be calculated from more fundamental properties.

In reality, the stimulus and response are usually direction dependent or *anisotropic* and hence are tensors, and therefore the material properties are also tensors of some rank. The equations mentioned above take the following look in tensorial notation:

$$P_\alpha = p_\alpha \Delta T, \quad M_\alpha = \chi_{\alpha\beta} H_\beta, \quad \sigma_{\alpha\beta} = c_{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta}$$

The physical property connecting a stimulus of rank-m to a response of rank-n will be of rank $(m+n)$. In the first equation listed, the stimulus ΔT is of rank zero and the response P_α is of rank one; hence the property p_α is of rank one, in the second equation, the stimulus H_β is of rank one and the response M_α is also of rank one; hence the property tensor $\chi_{\alpha\beta}$ is of rank two, in the last equation, the stimulus $\varepsilon_{\alpha\beta}$ is also of rank two and the response $\sigma_{\alpha\beta}$ is also of rank two; hence the property tensor $c_{\alpha\beta\gamma\delta}$ is of rank four. Few of These material properties in tensorial and usual representations, along with their corresponding stimulus, response, and ranks, are tabulated in *Table 3*.

As mentioned above, apart from the material properties, tensors are also used to describe fields and manifolds. Similar to property tensors, these tensors can be of various ranks. For example, the temperature field $T(x, y, z)$ is a scalar field, where each point in space is described by one number at that point. Hence, scalar fields are tensor fields of rank zero. On the other hand, electric and magnetic fields are vector fields or tensor fields of rank one, and their specification requires three numbers at each point in three-dimensional space. These three numbers are the components along the coordinate axes and give the direction and magnitude of the vector. The electromagnetic field tensor, introduced after the four-dimensional tensor formulation of special relativity in Minkowski space-time, is a second-rank tensor, and the electric and magnetic fields can be obtained from the components of the electromagnetic tensor.

Another example of a tensor field is the *Riemann curvature* of a manifold. A *manifold* is a topological space that locally resembles Euclidean space near each point. When distances and angles can be measured on the manifold, then it is called *Riemannian*.

Equation Name	Scalar/Vector Equation	Tensor Equation
Newton's 2nd Law	$m\frac{d\vec{v}}{dt}$	$m\frac{d\mu^{\mu}(\tau)}{d\tau}=f^{\mu}$
Maxwell Equations	$\nabla.\vec{B}=0$	$\partial_{\mu}(\frac{1}{2}\epsilon^{\mu\lambda\alpha\beta})F_{\alpha\beta}=0$
	$\nabla\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$	
	$\nabla.\vec{E}=4\pi\rho$	$\partial_{\beta}F^{\alpha\beta}=\frac{4\pi}{c}j^{\alpha}$
	$\nabla\times\vec{B}-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}=\frac{4\pi}{c}\vec{j}$	
Dirac Equation	$\left(\beta mc^2+c\sum_{n=1}^{3}\alpha_n p_n\right)\psi=0$	$(i\hbar\gamma^{\mu}\partial_{\mu}-mc)\psi=0$

TABLE 2. Few familiar undergraduate scalar/vector equations in Physics in their tensorial attire.

More distinctly, a *Riemannian manifold* is a differentiable manifold in which each tangent space is equipped with an inner product in a manner which varies smoothly from point to point. In tensor analysis, the *Riemann curvature tensor* is the most common way to express the curvature of Riemannian manifolds. It assigns a tensor to each point of a Riemannian manifold (i.e., it is a tensor field) that measures the extent to which the metric tensor is not locally isometric to that of Euclidean space. It is the sophistication or elegance of tensor analysis that it is able to capture the invariance aspect as well as the spatial peculiarities in one go. This, at once, can be seen from the general formula for the invariant line element in any space, $ds^2 = g_{ij}\,dx^i\,dx^j$, where g_{ij} is the metric tensor encoding the properties of the space.

4.3 Many System Quantum States

In quantum mechanics, a tensor product is used to describe a system that is made up of multiple quantum subsystems. The simple reason that tensor product is required to build the joint space is because the dimension of joint vector space of two separate quantum systems magnifies multiplicatively and not additively, and is equa to the product of dimensions of the two separate system vector spaces, i.e., $\dim(V\otimes V) = (\dim V)(\dim V)$. If V is the vector space of one system and V is the vector space of another system then the quantum state of both the systems is $V\otimes V$, where the symbol represents tensor product.

Now it is well known in tensor analysis that the rank of a tensor can be increased through the outer or tensor product. If we take Cartesian product of the two vector spaces $V\times V$ then the resultant dimension is just the direct sum of V and V i.e., $\dim(V\times V) = \dim V + \dim V$, because the vectors are ordered pairs of vectors $(V, V) \in V\times V$. The cartesian product space $V\times V$ is a space whose states are the states of system V or system V or both, whereas $V\otimes V$ is the vector space whose basic states are pairs of states, one from V and one from V. So the Cartesian product cannot account for a large Hilbert space constructed from the smaller sub Hilbert spaces but this large Hilbert space is accounted by tensor product space which is a much larger space than Cartesian space. Hence the tensor product is the fundamental building operation of quantum systems that occupies a central place in the subject of many body quantum mechanics.

Stimulus	Response	Property/ Coefficient	Scalar Equation	Tensor Equation	Tensor Rank
Temperature (ΔT)	Electric Polarization (P_α)	Pyroelectricity (p or p_α)	$P = p.\Delta T$	$P_\alpha = p_\alpha \Delta T$	First p_α
Temperature (ΔT)	Strain ($\varepsilon_{\alpha\beta}$)	Thermal expansion (a or $a_{\alpha\beta}$)	$\varepsilon = a.\Delta T$	$\varepsilon_{\alpha\beta} = a_{\alpha\beta} \Delta T$	Second $a_{\alpha\beta}$
Stress ($\sigma_{\beta\gamma}$)	Elec. Polarization (P_α)	Elec.Polarization (b or $b_{\alpha\beta\gamma}$)	$P = b.\sigma$	$P_\alpha = b_{\alpha\beta\gamma} \sigma_{\beta\gamma}$	Third $b_{\alpha\beta\gamma}$
Strain ($\varepsilon_{\gamma\delta}$)	Stress ($\sigma_{\alpha\beta}$)	Elasticity (c or $c_{\alpha\beta\gamma\delta}$)	$\sigma = c.\varepsilon$	$\sigma_{\alpha\beta} = c_{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta}$	Fourth $c_{\alpha\beta\gamma\delta}$

TABLE 3. Few material property tensor coefficients with their stimulus and response relation.

4.4 Entanglement

A major concept setting quantum reality apart from the classical one is the notion of entanglement. The fact that physical systems can be correlated in ways that exceed shared randomness of classically correlated systems is an important asset in the field of quantum computation. Quantum computation is a fast developing field, at the interface of Quantum mechanics, Computer science and Information theory, that tries to harness this weird aspect of quantum reality for technological purposes. Quantum computers are not limited to two states like present day computers which work by manipulating bits that exist in one of two states, $|0\rangle$ or $|1\rangle$. Quantum computers encode information as quantum bits or qubits that can exist in superposition of both $|0\rangle$ and $|1\rangle$ states at the same time. The tensor product succinctly captures this same distinct quantum behavior as has been discussed in the above section.

But if the states of two distinct quantum systems cannot be factorised as a tensor product of a wave function from one space with that from the other then the state is said to be "entangled". Thus the tensor product formalism comes into picture whenever entanglement is in consideration. The essence of quantum entanglement lies in the fact that there exist states in the tensor product space of physically separate systems that cannot be decomposed as tensor product of states from separate systems. In other words there exist states of the combined system that cannot be expressed in terms of definite states of the individual systems.

For example, suppose both V_A and V_B are two dimensional Hilbert spaces describing spin-1/2 degrees of freedom of two quantum particles. Each space can be spanned by an orthonormal basis of couple of states $|v_{Ai}\rangle \equiv |v_{Bi}\rangle = \{|0\rangle, |1\rangle\}$, representing spin-up and spin-down states. Then the tensor product space $V = V_A \otimes V_B$ is a four dimensional space spanned by pairwise basis of vectors drawn from V_A and V_B bases, that is $\{|v_{Ai}\rangle \otimes |v_{Bi}\rangle\} = \{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$. That is any state of the combined system is completely specified by $|v_{Ai}\rangle \otimes |v_{Bi}\rangle \in V_A \otimes V_B$. But as remarked above the opposite is not always true, that there exist states of the combined states which cannot be expressed as the tensor product of individual systems. For example consider the following quantum state of two spin 1/2 systems, $|\Phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$. There is no choice of basis that allows this state to be expressed in terms of separate individual particle states, i.e., $|\Phi\rangle \neq \frac{1}{\sqrt{2}}(|\Phi\rangle \otimes |\Phi\rangle)$. In such a case the two particles are in entangled state.

4.5 Data Storage and Mining

A big reason as to why tensors have now become ubiquitous in pure and applied sciences is because of the fact that they provide a means to organize multi-dimensional data, as tensors of higher rank can be represented as multi-dimensional arrays or matrices. Matrices are versatile objects that can be used for storing and accessing information in a systematic way. So in context of machine and deep learning tensors can be regarded as huge multi-dimensional containers having natural representation for data storage and data mining. It should be noted that though a **tensor** is often construed as a generalized matrix but every **matrix** is not a tensor. For instance any tensor of rank 2 can be cast in a matrix of order 2×2, but the converse that every matrix of 2×2 order is a rank-2 tensor is not true. The numerical values of the components of a tensor in its matrix representation depend on the transformation rules employed. The numerical values change when a transformation is made. Though the tensor remains invariant on coordinate transformation but its component do not and hence tensor despite appearing as an static entity are dynamical objects in essence. This dynamical aspect of tensors is what that distinguishes it from a mere matrix.

The *Figure 2* shows tensors of zero, first, second and third rank as single and multi-dimensional arrays or matrices. The availability of cheap and high computational power and storage devices has enabled extensive computations on vast amounts of data. Data mining is the process of extracting valuable knowledge or information from a large set of data. If tensor product is a useful operation for building large quantum states from sub quantum systems then tensor decomposition is a highly important tool for summarization and analysis of data. Much of the literature on data mining deals with tensor decomposition methods which are outside the scope of this article.

4.6 Non-Tensors

Finally, a brief note on non-tensors will be in order. Not every physical/mathematical quantity represented in symbols adorned with indices is a tensor. A trivial example is the components of a vector which is coordinate dependent. The individual components are not tensors because under vector transformation the components of the vector vary from system to system but in a way that the vector itself remains intact. Sometimes an entity is called a *qualified tensor* because it behaves as a tensor under a certain subclass of coordinate transformations. For instance, the differential element dx^i containing spatial components is a tensor in all the three dimensional Euclidean spaces but not in the four dimensional Minkowski space. Similarly the coordinates $\{x^i\}$ are not tensors as it makes no sense to add coordinates, for instance, adding the spherical coordinates of two points. In contrast to coordinates, the differentials dx^i are tensors. Also the Jacobian, $J^i_j = \left\{\frac{dx^i}{dx^j}\right\}$, used to quantify the changes in the infinitesimal lengths, areas, volumes, etc., that occur when changing the basis of a coordinate system, is also a non-tensor. This is because the Jacobian matrix is not defined *per se*, but is only defined with reference to two chosen coordinate systems. A tensor is an abstract entity which exists even if no basis has been referred to. The other obvious examples of non-tensors are the partial derivatives and also the Christoffel symbols. The partial/ordinary derivative does not in general yield a tensor because the derivative has no meaning outside the reference frame in which it is differentiated. For instance, if the derivative of a tensor in a coordinate system is zero, then it is not necessarily zero in other coordinate systems too. But the same does not hold for covariant derivatives, as the covariant derivatives are tensors, and if the covariant derivative vanishes in one frame, then it necessarily vanishes in all frames. Complete differentiation or covariant differentiation requires taking not only the component term but also the base vectors which are also spatially dependent, except in the case of orthonormal Cartesian coordinate system. The additional term that is added to the usual partial derivatives to make it covariant are called the Christoffel symbols. Not to mention the Christoffel symbols vanish in Cartesian coordinates.

5 Conclusions

Tensors are in essence abstract mathematical objects with deep and profound implications. Soon after the formulation of general theory of relativity the subject became popular and developed further rapidly in as much that it now appears in all branches of science and engineering. The key attribute of tensors is the facility to express invariance under appropriate transformation laws in either mathematical or physical properties/laws. But besides this fundamental property of invariance four other innate potentialities possessed by tensors comes in handy to express various aspects of physical reality in science namely anisotropicity, many particle quantum states, entanglement and big data storage capacity. Two extremely useful operations on tensors are tensor product and tensor decomposition. While tensor products are immensely relevant in quantum mechanics and quantum computation, tensor decomposition methods are highly needed in machine learning and deep learning. Therefore the bottom line is tensor is an incredibly important subject with fascinatingly significant features and amazingly wide implications.

Acknowledgments

The author acknowledges with gratitude the fruitful discussion with Arif Mohd (specialising in Classical and Quantum gravity), Visiting Assistant Professor, Physics Department, AMU, which helped improve the import of the article considerably.

References

[1] Jeevan Nadirji (2015).
An Introduction to Tensors and Group Theory, 2nd Edition (Birkhäuser).

[2] Sadri Hassani (1999).
Mathematical Physics – A Modern Introduction to Its Foundations, (Springer).

[3] Robert E. Newnham (2005).
Properties of Materials: Anisotropy, Symmetry, Structure (Oxford University Press).

[4] Zafar Ahsan (2005).
Tensors: Mathematics of Differential Geometry and Relativity (Prentice Hall of India).

[5] Albert Tarantola.
Tensors for Beginners,
`http://www.ipgp.fr/~tarantola/Files/Professional/Teaching/Diverse/TensorsForBeginners/Text/Tensors.pdf`

[6] Landsberg, J. M.
Tensors: Geometry and Applications, American Mathematical Society, Providence, Rhode Island.

[7] L. Cammoun et al. (2009).
Tensors in Image Processing and Computer Vision (Springer-Verlag London).

[8] Joseph C. Kolecki.
An Introduction to Tensors for Students of Physics and Engineering, National Aeronautics and Space Administration.
`https://www.grc.nasa.gov/www/k-2/Numbers/Math/documents/Tensors_TM2002211716.pdf`

A New Approach To Polynomial Algebra And Deformations

Abhijit Sen[1]

[1]Department of Physics, Suri Vidyasagar College, Suri,
Birbhum- 731101,West Bengal, INDIA
abhisen1973@gmail.com

Submitted on 21-01-2022

Abstract

Ordinary differential equations are expressed using a novel operator. A new type of polynomial algebra is introduced that is seen to occur in a wide number of ordinary differential equations. We use the novel approach to classify differential equations. Deformations of the underlying algebra are also investigated.

1 Introduction

A single variable Linear Differential Equation can be cast in the form [1]

$$[F(D) + P(x, \frac{d}{dx})]y(x) = Q(x) \tag{1}$$

where

$$F(D) = \sum_n a_n D^n \tag{2}$$

where

$D = x\frac{d}{dx}$ is a novel operator that we shall be using

$Q(x)$ gives the 'source term' of the differential equation

a_n are some x-independent constants and

P is an arbitrary operator/polynomial of x (denoted as $\sum_k P_k$, k=1,2,....) comprising of powers of x, derivatives with respect to x and other operators not expressible in the form $F(D)$. In general we indicate by P_k an operator that has net index 'k' (i.e the value by which the power of the variable is raised after the operator acts on a monomial). For instance x^4or $3x^5\frac{d}{dx}$ both are P_{+4} or P_4 terms. As a concrete example if we consider Hermite's differential equation

$$[x\frac{d}{dx} - n - \frac{1}{2}\frac{d^2}{dx^2}]H_n(x) = 0 \tag{3}$$

we can define

$F(D) = x\frac{d}{dx} - n = D - n$ and
$P(x, \frac{d}{dx}) = -\frac{1}{2}\frac{d^2}{dx^2}$.

1.1 A Specific Case

Let us concentrate on a specific case of the type of polynomial algebra where the exponent of the monomial is changed at the most by unity, Hence we need to have three type of terms

P_+ in which the exponent is increased by +1

P_0 in which the exponent is left unaltered and

P_{-1} in which the exponent is decreased by +1

We denote the algebra as $\{P_+, P_-, P_0\}$.

The rationale for restricting ourselves to this kind of algebra is that most differential equations of physical systems are of at most second order.

The polynomial algebra is defined by the two relations:

$$\begin{aligned} [P_0, P_\pm] &= \pm P_\pm \\ [P_+, P_-] &= f(P_0) \end{aligned} \tag{4}$$

Here $f(P_0)$ is a polynomial in P_0 and $P_\pm$ denote terms in P that increase or decrease the power of x by unity. We can obtain a second function $g(P_0)$ such that

$$f(P_0) = g(P_0) - g(P_0 - 1) \tag{5}$$

This equation clearly suggests that $g(P_0)$ has order one higher than $f(P_0)$. As we shall see, this relation is fundamental in defining the order of the algebra. In view of differential equations and the present algebra that we just postulated, let us put forward a new algebra that we see in certain classes of differential equations.

1.2 The New Algebra: $\{P_+, P_-, P_0\}$

We now focus on the new type of algebra that we had mentioned. It occurs in some specific differential equations. As we shall see this type of algebra occurs quite naturally for some well known class of differential equations (like the Heun class).

We consider the solutions to differential equations to be in the form of monomials. By monomials we mean an algebraic expression that has only one term. Some common examples of monomials are x^4, $3x^2$, $5x^4y^9$ etc.

Let the differential equation that we are studying be of the form:

$$\left[f_1(x)\frac{d^2}{dx^2} + f_2(x)\frac{d}{dx} + f_3(x)\right] y(x) = 0 \tag{6}$$

Keeping in mind the underlying algebra of the operators we have to take the $f_i(x)$ as

$$\begin{aligned} f_1(x) &= a_0x^3 + a_1x^2 + a_2x + a_3 \\ f_2(x) &= a_4x^2 + a_5x + a_6 \\ f_3(x) &= a_7x + a_8 \end{aligned} \tag{7}$$

which needs to be farther restricted to avoid 'unwanted' terms. In order to have the specific algebra and no other term, we must impose $a_3 = 0$. This gives three types of terms

$$\begin{aligned} P_+ &= a_0x^3\tfrac{d^2}{dx^2} + a_4x^2\tfrac{d}{dx} + a_7x \\ P_0 = F(D) &= a_1x^2\tfrac{d^2}{dx^2} + a_5x\tfrac{d}{dx} + a_8 \\ P_- &= a_2x\tfrac{d^2}{dx^2} + a_6\tfrac{d}{dx} \end{aligned} \tag{8}$$

We rewrite 8 using

$$P_0 = x\frac{d}{dx} = D \tag{9}$$

Rewriting $P_\pm$ in terms of P_0 we obtain

$$[P_+, P_-] = \alpha_1 P_0^3 + \beta_1 P_0^2 + \gamma_1 P_0 + \delta_1 = f(P_0) \tag{10}$$

A Casimir operator is defined as an operator that commutes with all the generators. It is defined by

$$C = P_- P_+ + g(P_0) \tag{11}$$

Explicitly working out the Casimir operator it comes out to be

$$C = a_6 a_7 + e \tag{12}$$

We obtain

$$\begin{aligned} \alpha_1 &= -4a_0a_2 \\ \beta_1 &= 6a_0a_2 - 3a_2a_4 - 3a_0a_6 \\ \gamma_1 &= (-2a_0a_2 + 3a_0a_6 - 2a_4a_6 - 2a_2a_7 + a_2a_4) \\ \delta_1 &= -a_6a_7 \end{aligned} \tag{13}$$

2 Deformations of Polynomial Algebras

As we saw in the last Section, general second order differential equations yield an algebra of $\{P_+, P_0, P_-\}$ type under specific conditions. In the present section we shall delve deeper into this type of algebra.

We now restrict ourselves to differential equations of the form

$$[P_+ + P_0 + P_-]\, y(x) = 0 \tag{14}$$

Here $g(P_0) = g(D)$ is such that

$$f(D) = g(D) - g(D-1) \tag{15}$$

where

$$f(D) = [P_+, P_-] \tag{16}$$

It is hence clear that $g(D)$ must have an order one higher than that of $f(D)$.

When $P_+$and P_-do not commute we get $f(D) \neq 0$ and we say that the algebra is deformed. Depending upon the highest power of D we define the degree of deformation of the algebra.

For instance, if we obtain

$$[P_+, P_-] = f(P_0) = aP_0^2 + bP_o + c \tag{17}$$

where a, b, c are constants, we say that it is a quadratic deformation of the algebra.

Since $f(D)$ is quadratic in D, clearly $g(D)$ must be cubic in D.

Taking $g(D) = \alpha D^3 + \beta D^2 + \gamma D + \delta$ we obtain upon comparison with 17 and inserting into 11

$$C = P_- P_+ + h_1(P_0) \tag{18}$$

where

$$h_1(P_0) = \frac{a}{3}P_0^3 + \frac{a+b}{2}P_0^2 + \frac{a+3b+6c}{6}P_0 + \delta \tag{19}$$

Substituting for $P_\pm$ we obtain the Casimir, though it is not of interest to us presently.

For the next higher order, if we obtain

$$[P_+, P_-] = f(P_0) = aP_0^3 + bP_o^2 + cP_0 + d \tag{20}$$

where a, b, c, d are constants, we say that it is a cubic deformation of the algebra.

As before, explicitly working out the Casimir one obtains

$$C = P_- P_+ + h_2(P_0)$$

where

$$h_2(P_0) = \frac{a}{4}P_0^4 + \left(\frac{b}{3} + \frac{a}{2}\right) P_0^3 + \left(\frac{a}{4} + \frac{b}{2} + \frac{c}{2}\right) P_0^2 + \left(\frac{b}{6} + \frac{c}{2} + d\right) P_0 \tag{21}$$

For higher order deformations we can explicitly work out following the same technique.

3 Conclusion

We find applications of the novel approach to differential equations in different cases like the Harmonic Oscillator [1], the Sextic Oscillator [1], the Heun differential equation[2][3] to name a few. The deformations that we have explored in this article seem to be novel results and possibly can be helpful in areas where algebraic structures of differential equations are dealt with.

References

[1] Solving Linear Differential Equations: A Novel Approach
N.Gurappa, A.Sen, R.Atre, P.K.Panigrahi
`www.arXiv.org/math.ph/1205.0385`

[2] Quasi-Exact Solvability and Deformations of Sl(2) Algebra
A.Roy, A.Sen, P.K.Panigrahi
`www.arXiv.org/math-ph/1304.2225`

[3] Exploring the Heun Class of Differential Equations
A.Sen
In preparation

Amplitude Modulation – A Simple Demonstration Experiment

Harjinderpal Singh Kalsi and Poonam Agarwal

Department of Physics,
Guru Nanak Khalsa College of Arts, Science & Commerce (Autonomous),
Matunga, Mumbai, India
kalsi.hs@gnkhalsa.edu.in
poonam.agarwal@gnkhalsa.edu.in

Submitted on 21-03-2022

Abstract

This article describes a very simple demonstration experiment of understanding Amplitude Modulation. Even though ICs are available for amplitude modulation, but this article shows that how one can understand the concept of amplitude modulation by using basic components and instruments available in any Undergraduate Physics laboratory.

1 Introduction

In Radio transmission, the audio signal (20 Hz to 20 kHz) is to be transmitted from a broadcasting station over a great distance to a receiver. However, the audio signal cannot be sent directly over the air for appreciable distance, even if the audio signal is converted into electrical signal, it cannot be sent very far without employing large amount of power. As, the radiation of electrical energy is practicable only at high frequencies, high frequency signals can be sent thousands of miles with small power. [1, 2]. Therefore, if an audio signal is to be transmitted to a longer distance, some methods must be devised which will allow transmission to occur at high frequencies and at the same time also carry the audio signal along with it. This is achieved by superimposing the low frequency electrical audio signal on a high frequency wave. The high frequency wave is called as the "Carrier" as it carries the low frequency audio signal. The resultant waves are known as modulated waves or radio waves and the process is called modulation as shown in Figure 1 [3, 4].

Modulation is the process of changing some characteristics (for example amplitude, frequency, phase) of the high frequency (carrier) wave in accordance with

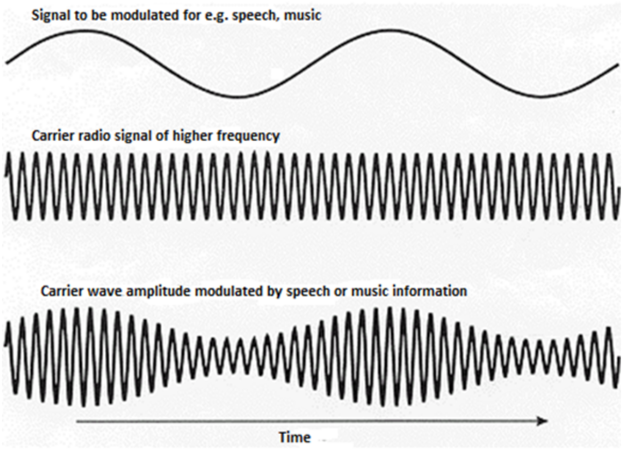

Figure 1: Modulation (Amplitude Modulation)

the intensity of the audio signal. Basically, there are three types of modulation:

- Amplitude modulation
- Frequency modulation
- Phase modulation

When the amplitude of high frequency carrier wave is changed in accordance with the intensity of the audio signal, it is called amplitude modulation as shown in Figure 1. Referring to Figure 1, in Amplitude Modulation only the amplitude of the high frequency carrier wave changes in accordance with the intensity of the audio signal, but the frequency of the modulated wave remains as that of the carrier waveform.

Whereas, when the frequency of the carrier wave is changed in accordance with the intensity of the audio signal, it is called Frequency Modulation (FM) as shown in Figure 2. In FM only the frequency of carrier wave is changed in accordance with the audio signal, however the amplitude of the carrier wave remains the same.

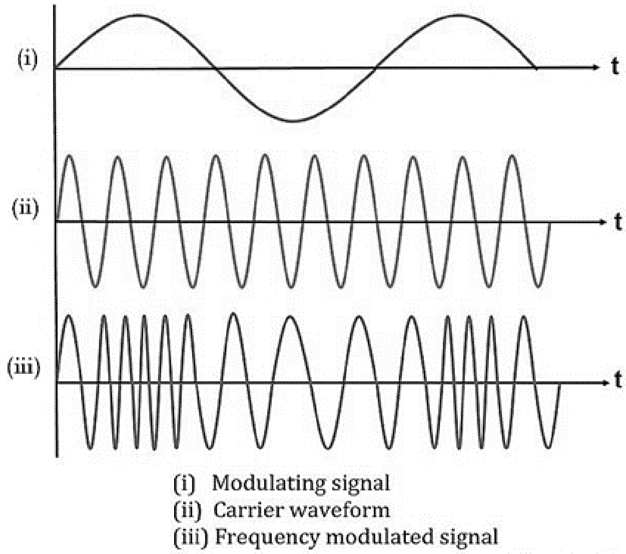

Figure 2: Frequency Modulation

2 The experimental set-up

In order to design a simple amplitude modulator circuit, we first design a basic Common Emitter Amplifier circuit having a certain voltage gain (Av) as shown in Figure 3. In the circuit, R_1 and R_2 are used as potential divider to apply a smaller ac input voltage to the amplifier. The input applied has a voltage of 50mV and frequency, f = 7.6kHz. The output observed is 180° out of phase with respect to the input signal as shown in Figure 4.

Now, we shall introduce another ac signal which is to be transmitted, to the emitter of the transistor as shown in Figure

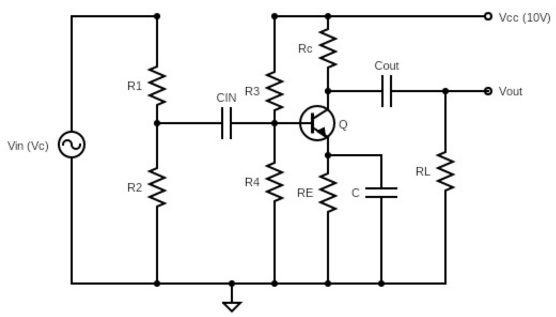

Figure 3: Common Emitter Amplifier (R_1= 10 kΩ, $R_2 = 1$ kΩ, R_3 = 22 kΩ, $R_4 = 10$ kΩ, R_C =15 kΩ, R_E =10 kΩ, $C_{in} = 0.1$ μF, $C_{out} = 0.01\mu$F, C = 1μF, Q = BC147).

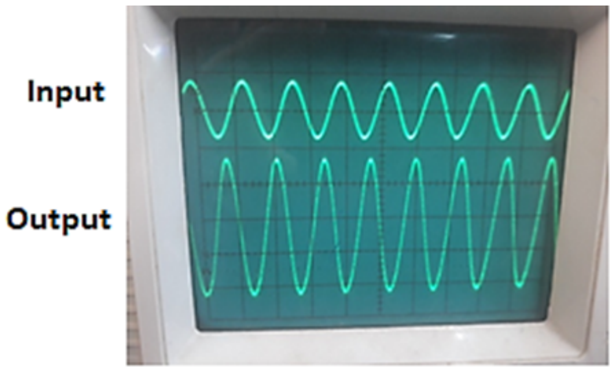

Figure 4: Input Output Waveform of CE Amplifier

5. This ac signal should be of low frequency. In this set-up we have kept 100Hz. Since this low frequency signal is now a part of the biasing circuit, hence it produces low frequency voltage variations in the emitter circuit.

As the voltage at the emitter varies, the Gain 'Av' of the amplifier also varies. Hence the amplitude of the amplified output varies in accordance with the intensity of the low frequency signal as shown in Figure 6a and

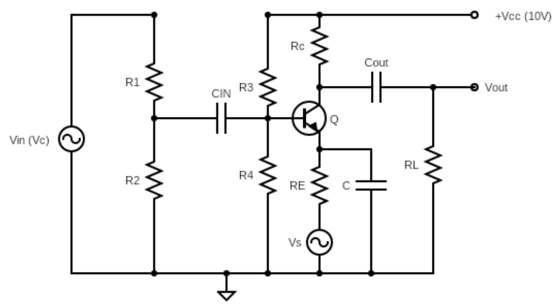

Figure 5: Low Frequency Signal applied to Emitter

6b. The output waveform observed at the Common Emitter amplifier looks like an amplitude modulated waveform, whereas the input signal applied to the input of the CE amplifier is the Carrier Waveform and the low frequency signal applied to the emitter of the transistor is the audio or modulating signal. So, when the amplitude of high frequency carrier wave is changed in accordance with the intensity of the audio signal it is called amplitude modulation. The frequency of the modulated signal is same as that of the Carrier frequency as shown in Figure 7.

One can also study the Modulation factor, which describes the depth of modulation i.e., the extent to which the amplitude of the carrier wave is changed by the audio signal. This can be done by varying the amplitude of the low frequency modulating signal. When the amplitude of the modulating signal (low frequency signal) is less than the amplitude of the high frequency carrier signal then it results in 'Under Modulation'.

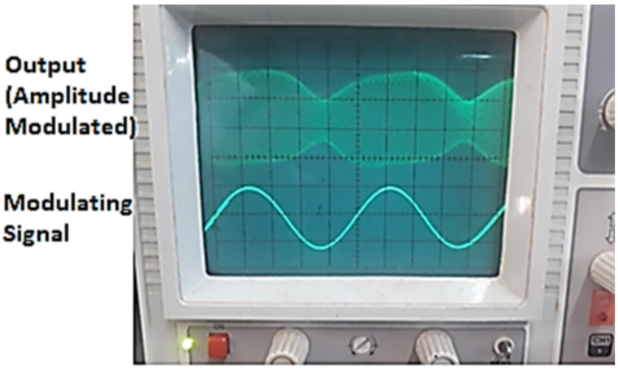

Figure 6a: Modulated Waveform and Modulating Signal

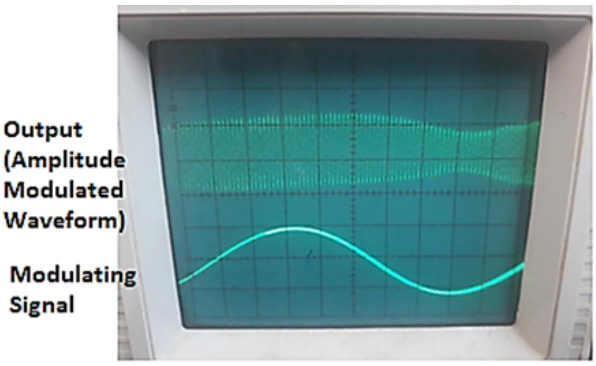

Figure 6b: Modulated Waveform and Modulating Signal

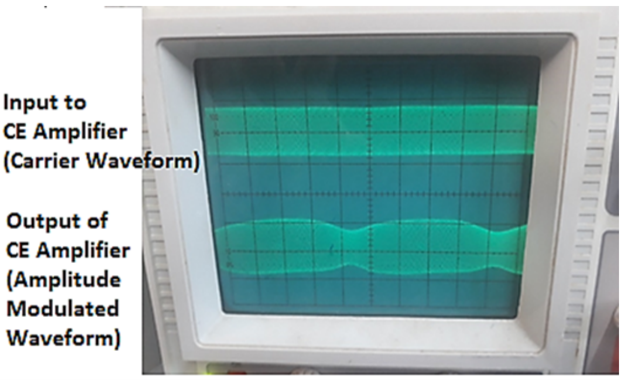

Figure 7: Frequency of Modulated Waveform same as Carrier Waveform

When the amplitude of the modulating signal (low frequency signal) is more than the amplitude of the high frequency carrier signal then it results in 'Over Modulation' as shown in Figure 8. The entire experimental set-up is shown in Figure 9.

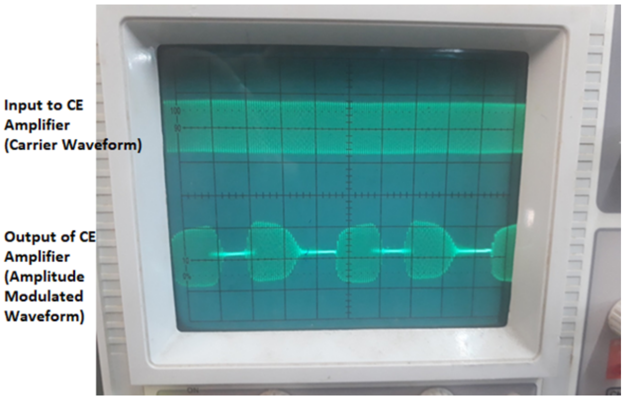

Figure 8: Over- Modulation

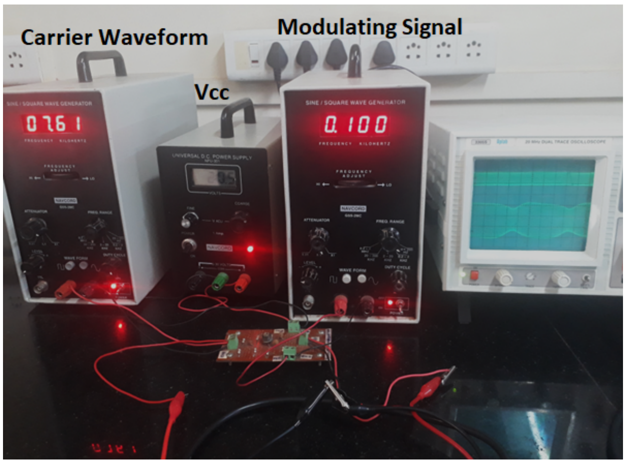

Figure 9: Entire Experimental Set-up

References

[1] George Kennedy and Bernard Davis, Electronic Communication Systems, 4th Edition, Tata McGraw Hill Education Private Limited.

[2] V. K. Mehta and Rohit Mehta, Principles of Electronics, S Chand Publishing House.

[3] Albert Malvino and David Bates, Electronic Principles, 8th Edition, Tata McGraw Hill Education Private Limited.

[4] Robert L. Boylestad and Louis Nashelsky, Electronic Devices and Circuit Theory, 11th Edition, Pearson