A semi-analytic formula of period for large-angle oscillations of a pendulum

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Abstract

Derivation of approximate formula for time-period of a simple pendulum for its large-angle oscillations has been carried out using polar co-ordinate $(r,\ \theta)$. The validity of the formula has been checked by comparing its results with the exact calculations obtained from elliptic integrals. An experiment was also carried out using the readily available apparatus in a general laboratory for amplitudes $\alpha \leq 80°$.

1 Introduction

The simple harmonic motion of a simple pendulum is introduced at high school level or at first year undergraduate level. The experiment is quite straightforward and is widely used to measure the value of acceleration due to gravity g at any given place. In these experiments, students are asked to keep angular amplitudes α as low as possible, normally, they are asked to maintain $\alpha < 5°$. The time-period T_0 is given by the relation $T_0 = \sqrt{l/g}$, where l is the length of the pendulum. The motion of pendulum is linear for small-oscillations. But, for large-angle oscillations, the motion becomes non-linear and hence there exists no simple formula for time-period T in this regime. So, students are rarely asked to perform large-amplitude oscillations.The exact value of period at different angular amplitudes can be estimated by evaluating elliptic integrals. However, the calculations are reasonably complicated for high school or first-year undergraduate students. So, it was felt by many authors that a simple linearized formula for the expression of time-period as a function of amplitude might encourage students to undertake large-amplitude experiments. Consequently, a number of such approximate formulae are available in [1, 2, 3, 4, 5, 6, 7, 8, 9] the literature. The easiest known formula for large-angle period is attributed to Bernoulli [1] and is given by $T = T_0(1+\alpha^2)/16$, where α is amplitude of oscillation. This formula is not suitable for a $\alpha > 40°$. Onestriking formula is due to Molina [3], which yields results within 1% error for $\alpha < 112°$ and it is given by $T = T_0(sin\alpha/\alpha)^{-\frac{3}{8}}$. Kidd and Fogg [4] obtained an appealing formula $T = T_0/\sqrt{cos(\alpha/2)}$, which agrees to within 1% for $\alpha < 100°$ with the exact values obtained from elliptic-integrals.

There have been a host of approximate formulae dedicated to large-angle regime. Some of these include the works of Santarelli *et al.* [2], Lima and Arun [6], Beléndez *et al.* [7], Butikov [8] and Big-Alabo [9].The derivation of most of these formulae has been made through linearization/approximations/modifications. Many of these are in good agreement with the exact elliptic-integral results with their own limitations.

In the present work, we endeavor to derive a formula based on an entirely new motivation.We use the basic concepts of radial and transverse accelerations of a body moving in a plane in terms of polar co-ordinates(r, θ). But, it turned out that the formula we derived was, interestingly, the same as derived by Kidd and Fogg [4]. We hope that the present paper will give students an alternative way of scrutinizing the derivation of the formula. We have presented the discourse in a more reasonable way and it is expected to be within the reach of their pedagogical understanding.

2 Derivation of formula for time-period

In terms of polar co-ordinates (r, θ), radial and transverse accelerations of a body moving in a plane are given by $(\ddot{r} - r\dot{\theta}^2)\hat{r}$ and $(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$, respectively. We take the point of suspension O of the pendulum as the pole (origin) and the downward vertical straight line from the pole O as the polar axis. With this orientation of the reference frame, angle θ represents instantaneous angular displacement from the equilibrium position and angle α represents the amplitude.

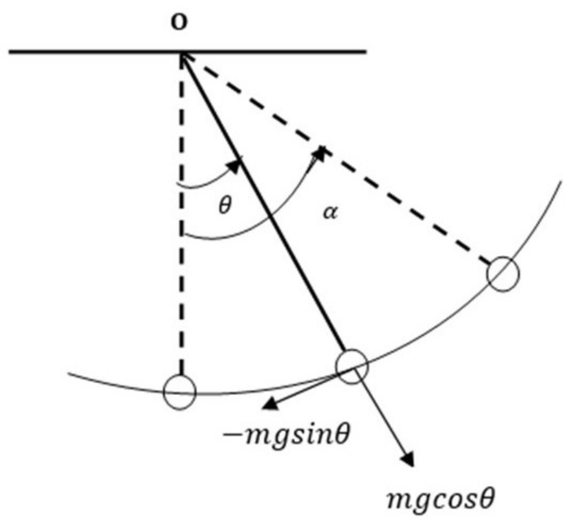

Figure 1: A schematic diagram showing the oscillation of a pendulum with amplitude α. The dashed lines denote $\theta = 0$ and $\theta = \alpha$ positions while the solid line represent an arbitrary intermediate position.

It is evident from Fig. 1 that the tension on the string, which acts radially inward providing the necessary centripetal force, is equal to $-mgcos\theta$. Similarly, $-mgsin\theta$ is the restoring force and it acts along the tangent to the trajectory of the pendulum.
Then, the equations of motion can written as

$$m(\ddot{r} - r\dot{\theta}^2) = -mgcos\theta \tag{1}$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = -mgsin\theta \tag{2}$$

Here, $r = l$, $\dot{r} = 0$ and $\ddot{r} = 0$. So, the above equations take the forms

$$\dot{\theta} = \sqrt{\frac{g}{l}}\sqrt{cos\theta}, \tag{3}$$

$$\ddot{\theta} = -(\frac{g}{l})sin\theta. \tag{4}$$

Eq. 4 can also be written as

$$\ddot{\theta} + \omega_0^2 sin\theta = 0, \tag{5}$$

where, $\omega_0 = \sqrt{g/l}$.

For small θ, $sin\theta \approx \theta$ and Eq. (5) becomes linear and it can be expressed as $\ddot{\theta} + \omega_0^2\theta = 0$, where, $\omega_0 = \sqrt{g/l}$ is the angular frequency and $T_0 = 2\pi\sqrt{l/g}$ the time-period for small oscillations. Also, for small θ, $cos\theta \approx 1$ and Eq. (3) yields $\dot{\theta} = \sqrt{\omega_0}$. Thus for small θ, Eqs. (3) and (4) are consistent with each other.
For large angle oscillations, it is obvious from Eq. (3) that the instantaneous angular frequency $\dot{\theta}$ is a function of angular-displacement θ and therefore, it is natural to ask as to what should be the expression for $\dot{\theta}$ when the pendulum is oscillating with an angular-amplitude α. The straightforward idea would be to replace the displacement θ by the amplitude α yielding $\dot{\theta} = \sqrt{g/l}\sqrt{cos\aleph}$. However, this would lead us to a wrong value of angular frequency as α is the maximum value of θ whereas it takes all the values lying between 0 and α during a swing. So, a simple logic to look for an appropriate replacement of θ, for linearization of Eq. (3), would be to find its average value between 0 and α, which is $\alpha/2$. With this replacement, expression for angular frequency ω in the large-amplitude regime can be obtained from Eq. (3) as $\omega = (\sqrt{g/l}\sqrt{cos(\alpha/2)})$. With $\omega_0 = \sqrt{g/l}$, the above equation takes the form $\omega = \omega_0\sqrt{cos(\alpha/2)}$, and, therefore, the time-period $T = 2\pi/\omega$ can be expressed as

$$T = \frac{T_0}{\sqrt{cos(\alpha/2)}}, \tag{6}$$

The above formula was first proposed by Kidd and Fogg [4]. However,their treatment is based on a different motivation.They obtained the force constant k using the relation $k =\mid dF/ds \mid$, where the force $F = -mgsin\theta$ and the arc length $s = l\theta$. Then, one gets $k = mgcos\theta/l$ and $\omega = \sqrt{k/m} = (\sqrt{gcos\alpha/l})$ and $T = T_0/\sqrt{cos\theta}$. They evaluated the exact value of period T_α at large amplitude α using the elliptic integral

$$T_\alpha = \frac{2T_0}{\pi}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - sin^2(\alpha/2)sin^2\phi}} \tag{7}$$

and noticed that the exact value of period T_α at an amplitude α matches well (within 1 % for $0 \leq \alpha \leq \pi/2$) when θ was replaced by $\alpha/2$ in the relation $T = T_0/\sqrt{cos\theta}$. It should be noted that Eq. (6) gives finite value of T for $0 \leq \alpha < \pi$. It diverges only at amplitude $\alpha = \pi$. So, in principle, the formula should work for amplitudes much beyond 90°. But we will see later that the error rises tp 3 % at 120°.

Now, we also make an attempt to justify the use the term $\alpha/2$ in place of α in Eq. (6). We will see in the following section that the angular frequency ω comes naturally, with a minor linearization of a term, in the form of $\omega = \omega_0\sqrt{cos(\theta/2)}$. We use the relation $Sin\theta = 2sin(\theta/2)cos(\theta/2)$ to re-write the Eq. (4) in the form

$$\frac{\ddot{\theta}}{2} + (\frac{g}{l}cos\frac{\theta}{2})sin\frac{\theta}{2} = 0. \tag{8}$$

we can write Eq. (8) in the form

$$\frac{\ddot{\theta}}{2} + \omega^2 sin\frac{\theta}{2} = 0, \tag{9}$$

where $\omega^2 = (g/l)cos(\theta/2)$.

Eq. (9) is similar to Eq. (5) except for two facts. Firstly, the instantaneous displacement in Eq. (5) is θ while that in Eq. (9) is $\theta/2$. Secondly, the term ω^2 in Eq. (9) is not a constant but a variable and is dependent on displacement θ. If we replace the variable displacement θ by the constant amplitude α, then, ω^2 becomes a constant. Denoting this constant value of ω^2 by ω^{2_α}, we can write

$$\omega_\alpha^2 = \frac{g}{l}cos\frac{\alpha}{2}. \tag{10}$$

With this minor but rational modification, Eq. (9) can be written as

$$\frac{\ddot{\theta}}{2} + \omega_\alpha^2 sin\frac{\theta}{2} = 0. \tag{11}$$

In the small-angle regime, $sin(\theta/2) \approx \theta/2$, then Eq. (11) can be expressed as

$$\ddot{\theta} + \omega_0^2\theta = 0, \tag{12}$$

which clearly represents a linear harmonic oscillation with angular frequency ω_α and time-period $T = T_0/\sqrt{cos(\alpha/2)}$ which is same as Eq.(6). The derivations from Eq. (8) to (12), based on the trigonometric formula $2sin(\theta/2)cos(\theta/2)$, was carried out after a long deliberation. However, while searching the literature later, it was found that the formula $ain\theta = 2sin(\theta/2)cos(\theta/2)$ was employed by the author L. E. Millet [5] in explaining the use of the term $\alpha/2$ instead of α in Eq. (6), but their reasoning and treatment is slightly different from ours.

We computed both T from Eq. (6) and T_α from Eq. (7) and compared them for the range of amplitudes $0° \leq \alpha < 180°$ for evaluation of the exact period T_α we run a simple fortran code implementing Simpson's 1/3 rule for numerical integration. A good agreement was observed between the two over a wide range of amplitudes as can be seen from Fig. 2, where we have plotted the values of T and T_α as a functional amplitude α taking $T_0 = 1$. The difference between the two results is less than 0.001 % for

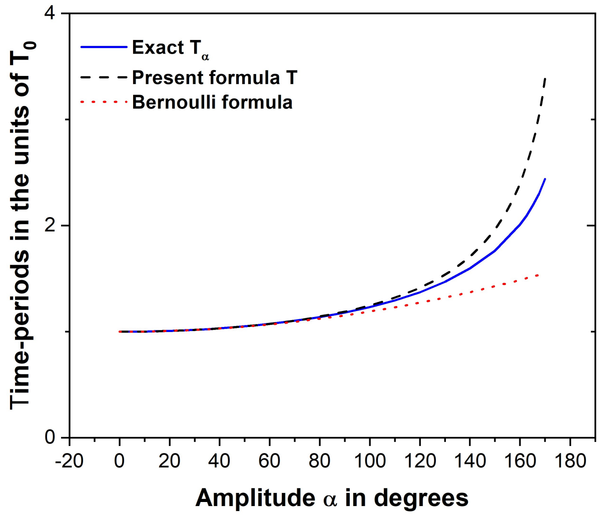

Figure 2: Curves showing variation of time-period (in the units of T_0) with angular amplitudes. The solid line in blue color represents the exact period T_α, the dotted line in red color represents the approximate period due to Bernoulli [1] and the dashed line in black color represents period T calculated from our formula of Eq. (6).

$\alpha \leq 20°$, it becomes 0.751 % at 90°, 3.011 % at 120°, about 10 % at 147°. However, the difference rises rapidly beyond 150° and it becomes as high as 174.4 % at 179°. It is also observed that the approximate period T is always higher than the exact period T_α.

3 Experiment

We performed an experiment with the readily available apparatus, namely a spherical-brass-bob of diameter 2.3 cm and an inextensible thread. Friction at the point of suspension was reduced by using two interlocked brass-rings having toroidal shapes. The experiments were carried out at a place

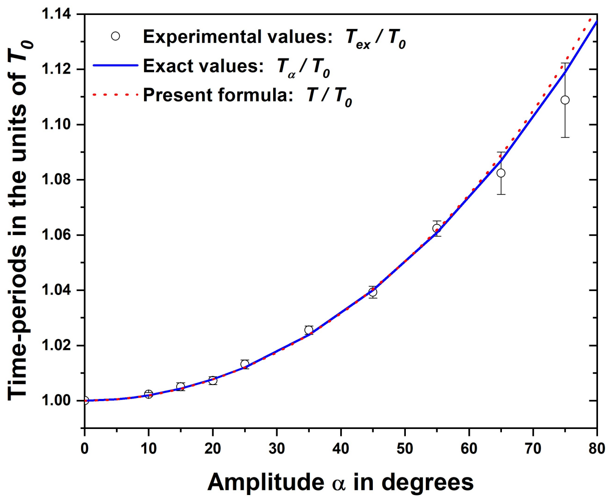

Figure 3: Curves showing variation of time-period (in the units of T_0) with amplitudes. The solid line in blue color represents the ratio T_α/T_0, where T_α is the exact period obtained from elliptic-integrals. The dashed line in red color represents the ratio T/T_0, where T is the period calculated from the present formula of Eq. (6) and the circles denote ratio of experimental data, T_{ex}/T_0.

having altitude $h = 350$ m from mean sea level. We calculated the value of acceleration due to gravity g at this place using the relation $g = g_0(1 + h/R_e)^(-2)$. Taking $g_0 = 9.8ms^(-2)$ and $R_e = 6378$ km, we obtained $g = 9.7989ms^(-2)$. The effective length l of the pendulum was 166.9 cm. We calculated the small-angle period T_0 using the relation $T_0 = 2\pi\sqrt{l/g}$ and found $T_0 = 2.5931$ s. Experimentally measured value of T_0 was found to be 2.5930 ± 0.0004 s and it is the weighted mean of several observations, we recorded up to 280 swings in one of the observations. The error in period for each observation was taken as $(0.2/N)$ s where N is total number of swings in that observation. The excellent agreement between the calculated and the measured value of T_0 showed that the pendulum was working within tolerable limits of air damping and friction at the peg.

It is well known that, at large-angle oscillations, one can get only few swings owing to large air damping. We could observe about 85 swings at angles between 5° and 10°, about 20 swings between 20° and 30° and only about 3 swings between 70° and 80°. So, it becomes difficult for human eyes to make precise measurements at higher angles. Therefore, we limited our observations at angles below 90°. The measurement at each amplitude is actually obtained by including the swings within a small range of amplitudes centered around that amplitude. For example, measurement at 35° is obtained by recording the time taken by all the possible number of swings between 30° and 40° The experimentally measured period T_{ex} at large amplitudes have been presented in Table 1. The values of T_{ex}/T_0, T_α/T_0 and T/T_0 are also presented in the table. The entries within brackets in the third row of Table 1 represent errors in the measurement of the ratios T_{ex}/T_0. The experimental results have also been depicted graphically in Fig. 3.

Table 1: Measured values of time-period T_{ex} and the ratios T_{ex}/T_0, T_α/T_0 and T/T_0 at different angular amplitudes. The figures within brackets in the third row represent errors in the measurement of the ratio T_{ex}/T_0.

Angles in degrees	10°	15°	20°	25°	35°	45°	55°	65°	75°
Period T_{ex} in s	2.5989 ±0.0016	2.6060 ±0.0036	2.6118 ±0.0037	2.6270 ±0.0042	2.6591 ±0.0039	2.6948 ±0.0056	2.7547 ±0.0072	2.8065 ±0.0200	2.8752 ±0.0348
T_{ex}/T_0	1.00229 (0.00061)	1.00503 (0.00138)	1.00725 (0.00143)	1.01313 (0.00161)	1.02549 (0.00149)	1.03927 (0.00215)	1.06235 (0.00279)	1.08234 (0.00771)	1.10881 (0.01243)
T_α/T_0	1.00191	1.00430	1.00767	1.001203	1.02383	1.03997	1.06083	1.08692	1.11896
T/T_0	1.00191	1.00431	1.00768	1.01207	1.02398	1.04038	1.06178	1.08889	1.12271

4 Conclusion

We derived a formula for time-period $T = T_0/\sqrt{cos(\alpha/2)}$ for large-angle oscillations of a simple pendulum. The formula was first derived by Kidd and Fogg [4] following a different line of motivation.The derivation is lucid and graspableby first year college students. The reason for the introduction of the term $\alpha/2$ in place of α has been elucidated qualitatively as well as analytically. The difference between our result and the exact value is negligible at smaller angles and it increases to 0.24 % (1.2 %) at 70° (100°). The difference rises to 10 % at 147°.We have also illustrated the possibility to carry out experiments to measure periods at amplitudes $\alpha \leq 80°$ with ordinary apparatus in a high school laboratory. For comparing such results the formula is sufficient. In fact, the experimental error occurring in an ordinary simple experiment, where times are recorded manually, would be higher than the difference between the exactperiod T_α and the approximate period T for $\alpha \leq 60°$.

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